ÜBUNGEN ZU

NUMERIK PARTIELLER DIFFERENTIALGLEICHUNGEN

für den 12. 10. 2005

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1. Show that a general linear second-order differential operator

$$Lu(x) \equiv -\bar{a}(x)u''(x) + \bar{b}(x)u'(x) + \bar{c}(x)u(x)$$

can always be written in the form

$$Lu(x) \equiv -\left(a(x)u'(x)\right)' + b(x)u'(x) + c(x)u(x)$$

and vice versa (under the assumption that the coefficients are sufficiently smooth).

2. Find variational formulations ($V = ?$, $V_0 = ?$, $V_g = ?$, $a = ?$, $F = ?$) for the following boundary value problems:

(a)

$$\begin{aligned} -u''(x) &= f(x) \quad x \in (0,1), \\ u(0) &= g_0, \\ u(1) &= g_1 \end{aligned}$$

(b)

$$\begin{aligned} -u''(x) &= f(x) \quad x \in (0,1), \\ -u'(0) &= g_0, \\ u'(1) &= g_1 \end{aligned}$$

(c)

$$\begin{aligned} -u''(x) &= f(x) \quad x \in (0,1), \\ u(0) &= g_0, \\ u'(1) &= \alpha_1(g_1 - u(1)) \end{aligned}$$

3. The scalar product in the Hilbert space $H = L^2(0,1)$ is given by

$$(w, v)_0 = \int_0^1 w(x)v(x)\,dx.$$

The scalar product in the Hilbert space $V = H^1(0,1)$ is given by

$$(w, v)_1 = (w, v)_0 + (w', v')_0 = \int_0^1 w(x)v(x)\,dx + \int_0^1 w'(x)v'(x)\,dx.$$

Let $f \in L^2(0,1)$. Consider the variational problem:

Find $u \in V$ such that

$$(u,v)_1 = (f,v)_0 \quad \text{for all } v \in V.$$

What is the corresponding boundary value problem in classical formulation?

4. Show for the variational formulation of the boundary value problem 2 (b):

   (a) If u is a solution, then, for any constant $c \in \mathbb{R}$, $u + c$ is also a solution.

   (b) If the boundary value problem has a solution u, then

   $$\langle F, c \rangle = 0$$

   for any constant function c. Hint: Choose the test function $v = c$.

5. Show the Poincaré inequality: There is a constant $C_P > 0$ with

$$\int_0^1 v(x)^2 \, dx \leq C_P^2 \left[\left(\int_0^1 v(x) \, dx \right)^2 + \int_0^1 v'(x)^2 \, dx \right] \quad \text{for all } v \in H^1(0,1).$$

   Hint: Prove the inequality for $v \in C^1[0,1]$ by integrating the identity:

$$v(y) = v(x) + \int_x^y v'(z) \, dz$$

   with respect to x over the interval $(0,1)$. Then use (without proof) that $C^1[0,1]$ is dense in $H^1(0,1)$.

6. Show only with the help of the Poincaré inequality:

$$u' \equiv 0 \Longleftrightarrow u \text{ is constant} \quad \text{for all } u \in H^1(0,1).$$

   Hint: Apply the Poincaré inequality for the function $v = u - \bar{u}$ with

$$\bar{u} = \int_0^1 u(x) \, dx.$$