4.2.4 Magnetostatics

- The magnetostatic problems simplifies to

$$(11)\begin{cases} (1)_A & \operatorname{curl} H = J_i \text{ with } \operatorname{div} J_i = 0, \ J_i - \text{given}, \\ (A)_{mg} & \operatorname{div} B = 0 \Rightarrow \exists A : B = \operatorname{curl} A \\ (1)_{BH} & B = \mu H + \mu_0 M \end{cases}$$

that leads to

$$(12)\quad \boxed{\begin{aligned} &\operatorname{curl} \nu \operatorname{curl} A = J_i - \operatorname{curl} M \text{ in } \Omega \\ &+ \text{IFC: } [A \times n] = 0 \text{ and } [\nu \operatorname{curl} A \times n] = 0 \text{ on } \Gamma_I \\ &+ \text{BC: } A \times n = 0 \text{ on } \Gamma_B \quad (\Rightarrow B \cdot n = 0!) \\ &\qquad \nu \operatorname{curl} A \times n = -J_s \text{ on } \Gamma_H \end{aligned}}$$

- For $\mu_r = 1$ (e.g. air), i.e. $\nu = \nu_0 := 1/\mu_0$, we have

$$J_i = \frac{1}{\mu_0} \nabla \times \nabla \times A = \frac{1}{\mu_0}\left(-\Delta A + \nabla(\nabla \cdot A)\right)$$

Coulomb gauge: $\boxed{\operatorname{div} A = 0}$

$$= -\frac{1}{\mu_0} \Delta A,$$

that means, with the Green function of $-\Delta$ (PDE), we obtain

$$A(y) = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{J_i(x)}{|y - x|} \, dx,$$

resulting in the Biot-Savartsche formula:

$$(13)\quad B(y) = \operatorname{curl}_y A(y) = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \nabla_y \times \frac{J_i(x)}{|y-x|} \, dx$$

$$= \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{y - x}{|y - x|} \times J_i(x) \, dx$$