JOHANNES KEPLER
UNIVERSITÄT LINZ

Bachelor Thesis

zur Erlangung des akademischen Grades Bachlorin in der Studienrichtung Technische Mathematik

Oceanic and Atmospheric Fluid Dynamics

Author:
Michaela Lehner

Supervisor:
Dr. Peter Gangl

28. Oktober 2017

Abstract

A system of partial differential equations called the primitive equations of the ocean and the atmosphere, describing the behaviour and the properties of those fluids, is fundamental when studying and predicting the behaviour of atmospheric and oceanic fluids. The most common field of application for the primitive equations is the modelling of the ocean and the atmosphere, as well as the description of their behaviour and predictions building on that. Many other models mathematically describing the ocean and the atmosphere are derived from the primitive equations, making them the basis of discussions concerning the ocean and the atmosphere. Goal of this thesis was the derivation of the primitive equations for the ocean and the atmosphere from the general physical equations and to consider boundary and initial conditions and some special settings.

Zusammenfassung

Fundamental beim Studieren und Vorhersagen des Verhaltens von atmosphärischen und ozeanischen Fluiden ist ein System partieller Differentialgleichungen, das das Verhalten und die Eigenschaften des Ozeans und der Atmosphäre genau beschreibt. Diese werden als die Primitiven Gleichungen des Ozeans und der Atmosphäre bezeichnet . Zugrunde liegen ihnen die physikalischen Eigenschaften, die diese kompressiblen Fluide ausmachen. Das wichtigste Anwendungsgebiet für diese Gleichungen ist die Modellierung des Ozeans und der Atmosphäre, sowie die Beschreibung ihres Verhaltens und der Vorhersagen, die aufgrund dessen getroffen werden können. Viele andere Modelle zur mathematischen Beschreibung des Ozeans und der Atmosphäre basieren auf den primitiven Gleichungen, wodurch sie zur Grundlage sämtlicher Betrachtungen des Ozeans und der Atmosphäre werden. Ziel dieser Arbeit war es, die Primitiven Gleichungen für den Ozean und die Atmosphäre aus den allgemeinen physikalischen Gesetzen herzuleiten, verschiedene Rand- und Anfangsbedingungen und einige Spezialfälle zu betrachten.

Acknowledgments

I would like to thank my advisor, Peter Gangl for the time he has invested in working with me on this project. My gratitude also goes to professor Ulrich Langer for teaching me the necessary basics in this field of study in his lecture and also for offering me a lot of support.

A big word of thanks finally goes to my mum, who always had my back when I had to focus on my studies and needed as little distraction as possible. Thanks for always supporting me!

Michaela Lehner
Linz, October 2017

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1 Introduction

The goal of this thesis is to derive a set of equations capable of describing the physical behaviour of atmospheric and oceanic fluids in a mathematical setting, while taking into account the numeric stability and the mathematical complexity of this set of equations.

When formulating models for the ocean and the atmosphere, we have to consider some physical properties first. Since the ocean and the atmosphere have so many fluid-dynamical properties in common, the study of one of them enriches our understanding of the other.

It is a well-established fact that the ocean can be seen as a slightly compressible fluid under Coriolis force. To describe the motions and states of the ocean, some basic quantities are necessary: the velocity field, the temperature, the salinity, the pressure and the density of the seawater. We will have a closer look into the origin of these quantities in Chapter 3.

The equations governing the behaviour of these quantities are the so-called general equations of a compressible fluid under Coriolis force, namely the momentum equation, the continuity equation, the thermodynamic equation, the equation of state and the equation of diffusion of the salinity.

After collecting a set of general equations, we derive the primitive equations for the ocean and the atmosphere. To do so, we consider the equations in spherical coordinates, since the earth can be approximated by a sphere. By applying the Boussinesq approximation which neglects density differences, and the hydrostatic approximation, which lets us ignore the z-coordinate, we can simplify our set of general equations to a set of primitive equations.

For special applications, especially observation of the long-term behaviour of the ocean and the atmosphere, we cannot simply neglect the vertical velocity, as it is done by the hydrostatic approximation. We will therefore also introduce a set of equation called the primitive equations with vertical velocity (PEV^2s), which are used especially for this field of application. For the atmosphere in particular, it is also useful to consider the primitive equations in the so-called p-coordinate-system, which has some advantages when it comes to calculations, reason being the continuity equation, which can in this coordinate setting be given in the form of an incompressible fluid.

In Chapters 2-4, to establish a mathematical and physical foundation behind models for the ocean and the atmosphere, we will derive the general equations of a compressible fluid, to be more specific, the thermodynamic equation and the Navier Stokes equations, and describe the physical effects of the Coriolis force and the diffusion of salinity. In Chapter 5, we will derive the so-called primitive equations, which will be used to model the ocean and the atmosphere. In Chapter 6, we will take into account some special settings and approximations. In Chapter 7, we will finally draw some conclusions.

2 Derivation of the Navier-Stokes Equations

2.1 Lagrangian and Eulerian Coordinates

We want to describe the motion of a velocity field during a time t under volume and surface forces. There are two different ways to describe the movement of a fluid (flow), namely the Eulerian and Lagrangian coordinates. While the Lagrangian framework describes a fluid particle though its location at a time, the Eulerian formulation focuses on one specific point in the space through which the fluid flows with a certain velocity, see e.g. [6].

2.1.1 Lagrangian Coordinates

Let $(t_S, t_E) \subset (T_1, T_2) \subset \mathbb{R}$ be our time interval for calculation and (T_1, T_2) be the time interval where we observe the movement of a fluid. Let $\Omega(t) \subset \mathbb{R}^3$ be the that is occupied by the fluid at time $t \in (T_1, T_2)$.
If $t_0 \in (T_1, T_2)$ is a fixed reference point, every fluid particle in $\Omega(t_0)$ can be identified by its coordinates $\boldsymbol{X} = (X_1, X_2, X_3) \in \Omega(t_0)$.
The velocity of the fluid in Lagrangian coordinates is then described by

$$\hat{v}(\boldsymbol{X}, t) = \frac{\partial \varphi}{\partial t}(\boldsymbol{X}, t), \tag{2.1}$$

where the vector function $\varphi(\boldsymbol{X}, t)$ is the trajectory of a specific fluid particle.

2.1.2 Eulerian Coordinates

Euler describes the movement of a fluid by its velocity field. $v(\boldsymbol{x}, t)$ shall be the velocity of a fluid particle in Eulerian coordinates, where we get with
$(\boldsymbol{x}, t) \in \{(\boldsymbol{x}, t) \in \mathbb{R}^4 : \boldsymbol{x} = (x_1, x_2, x_3) \in \Omega(t), t \in (T_1, T_2)\}$:

$$v(\boldsymbol{x}, t) = \hat{v}(\boldsymbol{X}, t) = \frac{\partial \varphi}{\partial t}(\boldsymbol{X}, t), \tag{2.2}$$

where $\boldsymbol{x} = \varphi(\boldsymbol{X}, t)$. It is more useful to use the Eulerian framework when handling fluids, since many physical factors are also given in Eulerian coordinates. Also, it is easier to make comments about the behaviour of a fluid when observing its velocity field.

2.2 Material Derivative

Special care has to be taken when time-dependent quantities have to be transformed between the two frameworks. In other words, we are interested in the total time derivative of the property of a piece of fluid. The total derivative is also known as material derivative and is derived as follows. See e.g. [6] and [7].

Given a velocity field $v(\boldsymbol{x}, t)$ and some scalar property $f(\boldsymbol{x}, t)$ for the fluid, which changes in time, we get with the help of the chain rule, the following definition:

Definition 2.1 (Material derivative)**.**

$$\frac{d}{dt} f(\boldsymbol{x}, t) := \frac{\partial}{\partial t} f(\boldsymbol{x}, t) + \nabla f(\boldsymbol{x}, t) \cdot v(\boldsymbol{x}, t). \tag{2.3}$$

2.3 Reynolds' Transport Theorem

Let $\omega(t) \in \Omega(t)$ be an arbitrary, sufficiently smooth, simply connected domain, which holds a fixed amount of fluid particles at time $t \in (T_1, T_2)$:

$$\omega(t) = \{\boldsymbol{x} = \varphi(\boldsymbol{X}, t) : \boldsymbol{X} \in \omega(t_0)\} \tag{2.4}$$

To deal with changes over time-dependent regions $\omega(t)$, a useful tool is the following theorem, as can also be seen in [6]:

Theorem 2.2 (Reynolds' Transport Theorem)**.** *Let $t_0 \in (T_1, T_2)$, $\omega(t_0)$ be a bounded, sufficiently smooth domain with $\overline{\omega(t_0)} \subset \Omega(t_0)$. $v : D \to \mathbb{R}^d$ and $F : D \to \mathbb{R}$ with $D := \left\{(\boldsymbol{x}, t) \in \mathbb{R}^{d+1} : \boldsymbol{x} \in \Omega(t), t \in (T_1, T_2)\right\}$ a C^1 scalar function.*
Then $\mathcal{F} := \int_{\omega(t)} F(\boldsymbol{x}, t)\, d\boldsymbol{x}$ is a well-defined C^1 function with

$$\frac{d\mathcal{F}}{dt}(t) = \int_{\omega(t)} \left[\frac{\partial F}{\partial t}(\boldsymbol{x}, t) + div\,(F \cdot v)(\boldsymbol{x}, t)\right] d\boldsymbol{x} \tag{2.5}$$

$$\frac{d}{dt} \int_{\omega(t)} F(\boldsymbol{x}, t)\, d\boldsymbol{x} = \int_{\omega(t)} \left[\frac{\partial F}{\partial t}(\boldsymbol{x}, t) + div\,(F \cdot v)(\boldsymbol{x}, t)\right] d\boldsymbol{x}, \tag{2.6}$$

where $div(F \cdot v) := \sum_{i=1}^{d} \frac{\partial(F \cdot v_i)}{\partial x_i} = \sum_{i=1}^{d} \frac{\partial F}{\partial x_i} v_i + F \cdot div\, v$.

Proof. For simplicity, we give the proof of formula (2.5) for the one-dimensional case only. If we apply the substitution rule for $d = 1$, it holds

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega(t)} F(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega(t_0)} F\left(\varphi(\boldsymbol{X},t),t\right)\frac{\partial\varphi}{\partial\boldsymbol{X}}\left(\boldsymbol{X},t\right)\,\mathrm{d}\boldsymbol{X} \\
&= \int_{\omega(t_0)} \frac{\partial}{\partial t}\left[F\left(\varphi(\boldsymbol{X},t),t\right)\frac{\partial\varphi}{\partial\boldsymbol{X}}\left(\boldsymbol{X},t\right)\right]\,\mathrm{d}\boldsymbol{X} \\
&= \int_{\omega(t_0)} \left(\frac{\partial F}{\partial t}(\varphi(\boldsymbol{X},t),t) + \frac{\partial F}{\partial\varphi}(\varphi(\boldsymbol{X},t),t)\frac{\partial\varphi}{\partial t}(\boldsymbol{X},t)\right)\frac{\partial\varphi}{\partial\boldsymbol{X}}(\boldsymbol{X},t)\,\mathrm{d}\boldsymbol{X} \\
&\quad + \int_{\omega(t_0)} F(\varphi(\boldsymbol{X},t),t)\frac{\partial}{\partial t}\frac{\partial\varphi}{\partial\boldsymbol{X}}(\boldsymbol{X},t)\,\mathrm{d}\boldsymbol{X} \\
&= \int_{\omega(t_0)} \left(\frac{\partial F}{\partial t}(\varphi(\boldsymbol{X},t),t) + \frac{\partial F}{\partial\varphi}(\varphi(\boldsymbol{X},t),t)\hat{v}(\boldsymbol{X},t)\right)\frac{\partial\varphi}{\partial\boldsymbol{X}}(\boldsymbol{X},t)\,\mathrm{d}\boldsymbol{X} \\
&\quad + \int_{\omega(t_0)} F(\varphi(\boldsymbol{X},t),t)\frac{\partial}{\partial\boldsymbol{X}}\hat{v}(\boldsymbol{X},t)\,\mathrm{d}\boldsymbol{X} \\
&= \int_{\omega(t)} \left(\frac{\partial F}{\partial t}(\boldsymbol{x},t) + \frac{\partial F}{\partial\boldsymbol{x}}(\boldsymbol{x},t)v(\boldsymbol{x},t)\right)\,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_{\omega(t)} F(\boldsymbol{x},t)\frac{\partial}{\partial\boldsymbol{x}}v(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\omega(t)} \left[\frac{\partial F}{\partial t}(\boldsymbol{x},t) + \frac{\partial F}{\partial x}(\boldsymbol{x},t)v(x,t) + F(\boldsymbol{x},t)\frac{\partial v}{\partial\boldsymbol{x}}(\boldsymbol{x},t)\right]\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\omega(t)} \left[\frac{\partial F}{\partial t}(\boldsymbol{x},t) + \frac{\partial}{\partial\boldsymbol{x}}(F\cdot v)(\boldsymbol{x},t)\right]\,\mathrm{d}\boldsymbol{x} \\
&= \int_{\omega(t)} \left[\frac{\partial F}{\partial t}(\boldsymbol{x},t) + \operatorname{div}\left(F\cdot v\right)(\boldsymbol{x},t)\right]\,\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

For the three-dimensional case, the proof can be found e.g. in [12].

Remark 2.3. *Applying Gauss' theorem, we can rewrite (2.5) in the following form:*

$$
\frac{d}{dt}\int_{\omega(t)} F(\boldsymbol{x},t)\,d\boldsymbol{x} = \int_{\omega(t)} \frac{\partial F}{\partial t}(\boldsymbol{x},t)\,d\boldsymbol{x} + \int_{\partial\omega(t)} F(\boldsymbol{x},t)(v(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x},t))\,ds_{\boldsymbol{x}} \tag{2.7}
$$

(2.8)

□

2.4 The Equations of Motion

2.4.1 Continuity Equation

With the help of Axiom 2.4 about the conservation of mass and Reynolds' Transport Theorem 2.2, we can derive the so-called continuity equation of a compressible fluid, which accounts for the flow of mass, see e.g. [6] and [7].

Axiom 2.4 (Conservation of mass). *Let $\omega(t) = \{\boldsymbol{x} = \varphi(\boldsymbol{X}, t) : \boldsymbol{X} \in \omega(t_0)\}$ be a region which holds a fixed amount of fluid particles at time $t \in (T_1, T_2)$. $\omega(t)$ will change its form, but (if it's divergence-free), not its mass.*

Since no mass can be created or destroyed, we get:

$$\frac{\partial \mathcal{M}}{\partial t}(t) = 0 \tag{2.9}$$

where $\mathcal{M}(t) = \int_{\omega(t)} \rho(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}$, and $\rho(\boldsymbol{x}, t)$ is the density of the fluid.

With Reynolds' transport theorem, we immediately get

$$0 = \frac{\partial \mathcal{M}}{\partial t}(t) = \frac{\partial}{\partial t}\int_{\omega(t)} \rho(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x} = \int_{\omega(t)} \left[\frac{\partial \rho}{\partial t}(\boldsymbol{x}, t) + \operatorname{div}(\rho \cdot v)(\boldsymbol{x}, t)\right] \mathrm{d}\boldsymbol{x} \tag{2.10}$$

for all $t \in (T_1, T_2)$ and for all bounded and sufficiently smooth domains with $\overline{\omega(t)} \subset \Omega(t_0)$.

From this, it follows that

$$\frac{\partial \rho}{\partial t}(\boldsymbol{x}, t) + \operatorname{div}(\rho \cdot v)(\boldsymbol{x}, t) = 0 \qquad \forall (\boldsymbol{x}, t) \in D \tag{2.11}$$

which is nothing but the continuity equation.

By means of $\sum_{i=1}^{3} \frac{\partial(\rho \cdot v_i)}{\partial x_i} = \sum_{i=1}^{3} \left[\frac{\partial \rho}{\partial x_i} v_i + \rho \frac{\partial v_i}{\partial x_i}\right]$, and with Definition 2.1 of the material derivative $\frac{\mathrm{d}\rho}{\mathrm{d}t} := \frac{\partial \rho}{\partial t} + \nabla \rho \cdot v = \frac{\partial \rho}{\partial t} + \rho \cdot \nabla v$, (2.11) is obviously equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(\boldsymbol{x}, t) + \rho(\boldsymbol{x}, t) \operatorname{div} v(\boldsymbol{x}, t) = 0 \tag{2.12}$$

2.4.2 Momentum Equation

To describe the behaviour of the momentum of a fluid responds to internal and imposed forces, we need the so-called momentum equation. See e.g. [4] for reference.
Following from the second and third law of Newton, we get the following axiom:

Axiom 2.5 (Conservation of Momentum)**.** *Changes of the momentum in time in a closed system of mass* $\omega(t) = \{\boldsymbol{x} = \varphi(\boldsymbol{X}, t) : \boldsymbol{X} \in \omega(t_0)\}$ *in time equals the forces acting on the system.*

$$\frac{d\mathcal{I}(t)}{dt} = F(\omega(t)), \tag{2.13}$$

where $\mathcal{I}(t) = \int_{\omega(t)} v(\boldsymbol{x}, t)\rho(\boldsymbol{x}, t)\, d\boldsymbol{x}$*.*

We call $F(\omega(t)) = F_B(\omega(t) + F_S(\omega(t))$ the forces acting on the system, which can be split into

$$\text{body forces on } \omega(t)\text{: } F_B(\omega(t)) = \int_{\omega(t)} \rho(\boldsymbol{x}, t) f(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x} \tag{2.14}$$

$$\text{surface forces on } \omega(t)\text{: } F_S(\omega(t)) = \int_{\partial\omega(t)} t^{(n)}(\boldsymbol{x}, t) \mathrm{d}s_{\boldsymbol{x}} \tag{2.15}$$

where $f(\boldsymbol{x}, t)$ is some external force acting on the system, and $t^{(n)}(\boldsymbol{x}, t)$ is the so-called total strain in a point $\boldsymbol{x} \in \partial\omega(t)$ at time t with normal vector $\boldsymbol{n}$.

The so-called *transformation formula* gives us the relationship

$$t^{n(x,t)}(\boldsymbol{x}, t) = \left[\sum_{j=1}^{3} \sigma_{ij}(\boldsymbol{x}, t)\boldsymbol{n}_j(\boldsymbol{x}, t)\right]_{i=\overline{1,3}} \tag{2.16}$$

between the stress tensor $\sigma(\boldsymbol{x}, t)$ and the total strain, as can also be seen in [6].

Applying Gauss' integration theorem to $F_S(\omega(t))$ provides

$$\begin{aligned} F_S(\omega(t)) &= \int_{\partial\omega(t)} t^{(n)}(\boldsymbol{x}, t) \mathrm{d}s_{\boldsymbol{x}} \\ &= \left[\int_{\partial\omega(t)} \sum_{j=1}^{3} \sigma_{ij}(\boldsymbol{x}, t)\boldsymbol{n}_j(\boldsymbol{x}, t) \mathrm{d}s_{\boldsymbol{x}}\right]_{i=\overline{1,3}} \\ &= \left[\int_{\omega(t)} \sum_{j=1}^{3} \frac{\partial\sigma_{ij}}{\partial x_j}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}\right]_{i=\overline{1,3}} \\ &= \int_{\omega(t)} \operatorname{div}\left(\sigma(\boldsymbol{x}, t)\right) \mathrm{d}\boldsymbol{x} \end{aligned} \tag{2.17}$$

From Axiom 2.5 and Theorem 2.2, we get for $i = \overline{1,3}$:

$$\begin{aligned} \frac{\mathrm{d}\mathcal{I}_i(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\omega(t)} v_i(\boldsymbol{x}, t)\rho(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x} \\ &= \int_{\omega(t)} \frac{\partial(v_i\rho)}{\partial t}(\boldsymbol{x}, t) + \operatorname{div}\left(v_i\rho \cdot v\right)(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{2.18}$$

Also, using our definition of the forces acting on a system ((2.14)-(2.15)), and the consecutive calculations, we get

$$F_i(\omega(t)) = \int_{\omega(t)} \rho(\boldsymbol{x},t) f_i(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} + \int_{\omega(t)} \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} \tag{2.19}$$

for $i = \overline{1,3}$.

Those two equations are equal. In conclusion, we can write the equation of motion in conservative form. For $i = \overline{1,3}$, it holds that:

$$\frac{\partial \rho v_i}{\partial t}(\boldsymbol{x},t) + \mathrm{div}\,(\rho v_i \cdot v)(\boldsymbol{x},t) = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}(\boldsymbol{x},t) + \rho(\boldsymbol{x},t) f_i(\boldsymbol{x},t) \tag{2.20}$$

Therefore, we get the equation of motion in convective form

$$\frac{\partial \rho}{\partial t} v_i + \rho \frac{\partial v_i}{\partial t} + \sum_{j=1}^{3} \left[\frac{\partial (\rho v_i)}{\partial x_j} v_j + \rho v_i \frac{\partial v_j}{\partial x_j} \right] = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}(\boldsymbol{x},t) + \rho f_i, \tag{2.21}$$

for $i = \overline{1,3}$. Using the continuity equation (2.11), we can transform (2.21) into the form

$$\rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v = \mathrm{div}\,\sigma + \rho f \tag{2.22}$$

on the whole domain $D = \left\{ (\boldsymbol{x},t) \in \mathbb{R}^{d+1} : \boldsymbol{x} \in \Omega(t), t \in (T_1, T_2) \right\}$.

By using the definition of the material derivative (2.3), this becomes

$$\rho \frac{\mathrm{d}v}{\mathrm{d}t} = \mathrm{div}\,\sigma + \rho f. \tag{2.23}$$

See also [4] for reference.

2.4.3 Navier Stokes Equations

Together, the momentum equation and the continuity equation are called the Navier Stokes Equations for a compressible fluid:

$$\rho \frac{\mathrm{d}v}{\mathrm{d}t} = \mathrm{div}\,\sigma + \rho f, \tag{2.24}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \mathrm{div}\, v = 0. \tag{2.25}$$

We still have to define the forces f in the momentum equation. This will be done in a separate chapter.

3 Derivation of the thermodynamic equations

3.1 Equation of State

Since the momentum and the continuity equation provide us with four equations (the momentum equation is nothing but 3 coupled PDEs), but five unknowns, it is necessary to add another equation to our set of equations describing oceanic and atmospheric fluids. The equation of state relates the various thermodynamic variables, temperature, pressure, composition (salinity) and density to each other, see e.g [4].
Generally speaking, the equation of state is the following:

$$\rho = f(T, S, p), \tag{3.1}$$

where ρ is the density, T the temperature, S the salinity and p the pressure.

It is necessary for our models to provide an explicit equation of state.

3.1.1 Ocean

When thinking about the behaviour of the ocean, it is natural to expect ρ to decrease if the temperature increases, and ρ to increase if the salinity increases. Therefore we can formulate a linear law for the ocean, where ρ_0, T_0 and S_0 are reference values of density, temperature and salinity, and β_T and β_S are constant expansion coefficients:

$$\rho = \rho_0(1 - \beta_T(T - T_0) + \beta_S(S - S_0)) \tag{3.2}$$

3.1.2 Atmosphere

We can consider the atmosphere as an ideal gas. Since the state of a gas can be determined by its temperature, pressure and volume, by reformulating the density in terms of pressure we get an equation of state in the form

$$p = R\rho T, \tag{3.3}$$

where $R = c_p - c_v$ is the specific gas constant, obtained by subtracting the specific heat at constant volume c_v from the specific heat at constant pressure c_p.

3.2 Thermodynamic Equations

In fluids where the equation of state involves temperature, the thermodynamic equation is necessary for obtaining a closed system of equations. See e.g. [4]

We consider the ocean and the atmosphere as systems in equilibrium. Therefore, we can express the internal energy I per unit mass of a system as function of the specific volume $\alpha = 1/\rho$, the specific entropy η and the chemical composition, parametrized as the salinity S:

$$I = I(\alpha, \eta, S), \tag{3.4}$$

It should be noted that for the atmosphere, we will neglect the chemical composition.

(3.4) is called the *fundamental equation of state*, for which the first differential formally gives

$$\mathrm{d}I = \frac{\partial I}{\partial \alpha}\,\mathrm{d}\alpha + \frac{\partial I}{\partial \eta}\,\mathrm{d}\eta + \frac{\partial I}{\partial S}\,\mathrm{d}S. \tag{3.5}$$

Basis for our derivation of the thermodynamic equation is the first law of thermodynamics, which states the principle of conservation of energy, which again tells us that the internal energy of a body might change due to work done, heat input or maybe changes in its chemical composition, while the total amount of energy will always stay the same.

Axiom 3.1 (First law of thermodynamics)**.** *The internal energy of an isolated system is constant:*

$$dI = dQ - dW + dC, \tag{3.6}$$

where dW is the work done by the body, dQ is the heat input to the body, dC denotes the changes in the chemical composition of the body (e.g. salinity), and dI is finally the change in internal energy per unit mass.

The work, heat, and chemical composition can change due to various reasons. We want to specify them according to our requirements:

The second law of thermodynamics tells us that heat in a system can only flow in one direction (e.g. from a warmer to a colder location), but not backwards. We can also formulate this in terms of entropy:

Axiom 3.2 (Second law of thermodynamics)**.** *Increase of the entropy times temperature results from a transfer of heat:*

$$T\,d\eta = dQ \tag{3.7}$$

The work done by a body equals the pressure p times the change in volume α:

$$\mathrm{d}W = p\,\mathrm{d}\alpha. \tag{3.8}$$

Changes in chemical composition can in the case of the ocean be related to the change in salinity $\mathrm{d}S$ times the chemical potential μ of the solution:

$$\mathrm{d}C = \mu\,\mathrm{d}S \tag{3.9}$$

In case of the atmosphere, we have of course no salinity, but changes in the amount of water in the air. These changes are brought about by changes in the temperature, which are already covered by the heat input. Therefore, for the atmosphere, we omit the changes in chemical composition.

Adding up all we know about the conversation of energy, the equations (3.5)-(3.9) result in the *fundamental thermodynamic relation*

$$\mathrm{d}I = T\,\mathrm{d}\eta - p\,\mathrm{d}\alpha + \mu\,\mathrm{d}S. \tag{3.10}$$

Assuming that locally the fluid is in thermodynamic equilibrium. Then the thermodynamic quantities (e.g. temperature, pressure, density,...) can vary in space, but locally they are related by the equation of state and Maxwell's relations.

3.2.1 Atmosphere

As we said before, we omit changes in chemical composition for the atmosphere, resulting in the thermodynamic relation

$$\mathrm{d}I = -p\,\mathrm{d}\alpha + \mathrm{d}Q \tag{3.11}$$

Since we can see the atmosphere as an ideal gas at constant pressure c_p and constant volume c_v, the internal energy is a function of temperature only: $\mathrm{d}I = c_v\,\mathrm{d}T$.

$$\mathrm{d}Q = c_v\,\mathrm{d}T + p\,\mathrm{d}\alpha \tag{3.12}$$

Using the relations $\alpha = RT/p$ and $c_p - c_v = R$, we get

$$\mathrm{d}Q = c_p\,\mathrm{d}T - \alpha\,\mathrm{d}p \tag{3.13}$$

By using the fact that, locally, the atmosphere is in thermodynamic equilibrium, we can form the material derivative:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = c_p\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}\frac{\mathrm{d}p}{\mathrm{d}t} \tag{3.14}$$

where $\frac{\mathrm{d}Q}{\mathrm{d}t}$ is called the heat flux per unit density in a unit time interval.

3.2.2 Ocean

Reformulating the fundamental thermodynamic relation (3.10) in terms of entropy leaves us with the equation

$$\begin{aligned} \mathrm{d}\eta &= \frac{1}{T}\left(\mathrm{d}I + p\,\mathrm{d}\alpha - \mu\,\mathrm{d}S\right) \\ &= \frac{1}{T}\left(c_v\,\mathrm{d}T + p\,\mathrm{d}\alpha - \mu\,\mathrm{d}S\right) \\ &= \frac{1}{T}\left(\mathrm{d}Q - \mu\,\mathrm{d}S\right) \end{aligned} \tag{3.15}$$

Which becomes, after forming the material derivative and with the help of (3.11), the so-called *entropy equation*:

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{1}{T}\left(\frac{\mathrm{d}Q}{\mathrm{d}t} - \mu\frac{\mathrm{d}S}{\mathrm{d}t}\right) \tag{3.16}$$

If we take the entropy to be a function of pressure, temperature and salinity, we get

$$\begin{aligned} T\,\mathrm{d}\eta &= T\left(\frac{\partial\eta}{\partial T}\right)_{p,S}\mathrm{d}T + T\left(\frac{\partial\eta}{\partial p}\right)_{T,S}\mathrm{d}p + T\left(\frac{\partial\eta}{\partial S}\right)_{T,p}\mathrm{d}S \\ &= c_p\,\mathrm{d}T + T\left(\frac{\partial\eta}{\partial p}\right)_{T,S}\mathrm{d}p + T\left(\frac{\partial\eta}{\partial S}\right)_{T,p}\mathrm{d}S \end{aligned} \tag{3.17}$$

From equation (3.16) and (3.17), we get:

$$\begin{aligned} c_p\frac{\mathrm{d}T}{\mathrm{d}t} + T\left(\frac{\partial\eta}{\partial p}\right)_{T,S}\frac{\mathrm{d}p}{\mathrm{d}t} + T\frac{\partial\eta}{\partial S}\frac{\mathrm{d}S}{\mathrm{d}t} &= \frac{\mathrm{d}Q}{\mathrm{d}t} - \mu\frac{\mathrm{d}S}{\mathrm{d}t} \\ \frac{\mathrm{d}T}{\mathrm{d}t} + \frac{T}{c_p}\left(\frac{\partial\eta}{\partial p}\right)_{T,S}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{T}{c_p}\frac{\partial\eta}{\partial S}\frac{\mathrm{d}S}{\mathrm{d}t} &= \frac{1}{c_p}\frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{\mu}{c_p}\frac{\mathrm{d}S}{\mathrm{d}t} \\ \frac{\mathrm{d}T}{\mathrm{d}t} + \frac{T}{c_p}\left(\frac{\partial\eta}{\partial p}\right)_{T,S}\frac{\mathrm{d}p}{\mathrm{d}t} &= \underbrace{\frac{1}{c_p}\frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{\mu}{c_p}\frac{\mathrm{d}S}{\mathrm{d}t} - \frac{T}{c_p}\frac{\partial\eta}{\partial S}\frac{\mathrm{d}S}{\mathrm{d}t}}_{=:Q_T} \end{aligned} \tag{3.18}$$

where we will call Q_T the heat diffusion.

For further calculations, we will need Maxwell's relations.

Theorem 3.3 (third of Maxwell's Relations)**.**

$$\left(\frac{\partial\eta}{\partial p}\right)_T = -\left(\frac{\partial\alpha}{\partial T}\right)_p \tag{3.19}$$

Proof. Due to (3.7) and (3.11), We can write

$$\begin{aligned} \mathrm{d}I &= T\,\mathrm{d}\eta - p\,\mathrm{d}\alpha = \mathrm{d}(T\,\mathrm{d}\eta) - \eta\,\mathrm{d}T - \mathrm{d}(p\alpha) + \alpha\,\mathrm{d}p, \\ \mathrm{d}G &= -\eta\,\mathrm{d}T + \alpha\,\mathrm{d}p, \end{aligned}$$

where G is the Gibbs function. Formally, we have now

$$\mathrm{d}G = \left(\frac{\partial G}{\partial T}\right)_p \mathrm{d}T + \left(\frac{\partial G}{\partial p}\right)_T \mathrm{d}P.$$

From the last two equations, we see that $\eta = -(\partial G/\partial T)_p$ and $\alpha = (\partial G/\partial p)_r$. Since

$$\frac{\partial^2 G}{\partial p \partial T} = \frac{\partial^2 G}{\partial t \partial p}$$

it holds that

$$\left(\frac{\partial \eta}{\partial p}\right)_T = -\left(\frac{\partial \alpha}{\partial T}\right)_p$$

□

Using this theorem, we get

$$\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{T}{c_p}\left(\frac{\partial \alpha}{\partial T}\right)_p \frac{\mathrm{d}p}{\mathrm{d}t} = Q_T \tag{3.20}$$

Since density and temperature can be related through a measurable coefficient of thermal expansion β_T, we get:

$$\left(\frac{\partial \alpha}{\partial T}\right)_p = \frac{\beta_T}{\rho} \tag{3.21}$$

With this, the thermodynamic equation gets the form

$$\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\beta_T T}{c_p \rho}\frac{\mathrm{d}p}{\mathrm{d}t} = Q_T \tag{3.22}$$

Because liquids are characterized by a small thermal expansion coefficient, it is sometimes acceptable to neglect the second term on the left-hand side of equation(3.22), resulting in the thermodynamic equation for the ocean

$$\frac{\mathrm{d}T}{\mathrm{d}t} = Q_T \tag{3.23}$$

Remark 3.4. *The entropy equation and the internal energy equation are equivalently connected via the equation of state. Both equations are usually referred to as "the thermodynamic equation"*

All calculations in this chapter can also be found in [4].

4 Description of physical effects

4.1 Forces on the momentum equation

The forces in the momentum equation consist of several physical factors, namely the pressure gradient, the gravity, the Coriolis force and the dissipative force. For further information, see [4] and [5]

4.1.1 Pressure Force

As stated in [4], when describing a fluid, pressure is the normal force per unit area within or at the boundary of the fluid. We call F the pressure force per unit value.

The pressure force on a domain of the fluid is the integral of the pressure over its boundary, in other words

$$F_p = -\int_{\partial w(t)} p \cdot \boldsymbol{n} \, \mathrm{d}s_{\boldsymbol{x}} \tag{4.1}$$

The minus-sign comes from the direction of the pressure, which is pointed inwards, while the normal vector points outwards.

Applying the divergence theorem we get that

$$F_p = -\int_{w(t)} \nabla p \; \mathrm{d}\boldsymbol{x} \tag{4.2}$$

The pressure force per unit volume can therefore be called the pressure gradient, defined as

$$F = -\nabla p \tag{4.3}$$

4.1.2 Gravity

Gravity also has to be taken into account when dealing with atmospheric and oceanic fluids. It is indicated by

$$\boldsymbol{g} \approx (0, 0, -9.8) \qquad \left[\frac{m}{s^2}\right] \tag{4.4}$$

4.1.3 Coriolis Force

Since the ocean and the atmosphere can be seen as layers of fluid on a sphere, their motion is influenced by rotation. We can include the effects of the rotation into our equations by using the so-called Coriolis force, see e.g. [4]

Rate of change of a vector

To consider rotating fluids, we have to relate the rate of change of a vector in the inertial and rotating frames. Following the steps in [4], we get a relation that we only want to define here:

Definition 4.1 (Rate of change of a vector)**.** *Let B be the a vector that changes in the inertial frame I. The rates of change in the inertial frame and the rotating frame R can be related by*

$$\left(\frac{dB}{dt}\right)_I = \left(\frac{dB}{dt}\right)_R + \Omega \times B \tag{4.5}$$

where we call Ω the angular velocity or rotation rate.

Remark 4.2. *The rotation rate Ω is not to be confused with the domain $\Omega(t)$ we used in Chapter 2 to define the region occupied by the fluid at time t.*

Velocity in a rotating frame

By applying (4.5) on the position of a particle $\boldsymbol{x} = \varphi(\boldsymbol{X}, t)$, we get the relation

$$v_I = v_R + \Omega \times \boldsymbol{x} \tag{4.6}$$

If we apply (4.5) in terms of v_R on the above equation, we get

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(v_I - \Omega \times \boldsymbol{x})\right)_I = \left(\frac{\mathrm{d}v_R}{\mathrm{d}t}\right)_R + \Omega \times v_R \tag{4.7}$$

$$\left(\frac{\mathrm{d}v_I}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}v_R}{\mathrm{d}t}\right)_R + \Omega \times v_R + \frac{\mathrm{d}\Omega}{\mathrm{d}t} \times \boldsymbol{x} + \Omega \times v_I \tag{4.8}$$

Under the assumption that the rotation rate Ω is constant, we get

$$\left(\frac{\mathrm{d}v_R}{\mathrm{d}t}\right)_R = \left(\frac{\mathrm{d}v_I}{\mathrm{d}t}\right)_I - 2\Omega \times v_R - \Omega \times (\Omega \times \boldsymbol{x}) \tag{4.9}$$

where $2\Omega \times v_R$ is the Coriolis force per unit mass and $\Omega \times (\Omega \times \boldsymbol{x})$ is the centrifugal force, which we will omit in our model. The Coriolis force is a so-called pseudo-force with basic properties:

- There is no Coriolis force on stationary bodies in the rotating frame
- The Coriols force deflects moving bodies at right angle to their travel direction
- The Coriolis force does not work on a body due to its perpendicularity to the velocity, therefore $v_R \cdot (\Omega \times v_R) = 0$.

4.1.4 Heat dissipation

The heat dissipation D is usually related to the stress tensor σ. See also [10] for reference.

$$D = -\frac{1}{\rho}\operatorname{div}\sigma \tag{4.10}$$

4.1.5 Momentum equation with physical effects

Applying pressure force, gravity, Coriolis force and heat dissipation on the momentum equation leaves us with the equation

$$\rho\frac{\mathrm{d}v_R}{\mathrm{d}t} + 2\rho\Omega \times v_R + \nabla p + \rho\boldsymbol{g} = D \tag{4.11}$$

with v_R being the velocity field on the rotating frame. We will hereafter call this velocity field in 3D $\boldsymbol{V}_3$, to match with the references [1] and [2].

4.2 Diffusion of Salinity

As given in [4], the diffusion of salinity represents the effects of vaporation and precipitation at the ocean surface, as well as molecular diffusion. It changes only when there are non-conservative sources and sinks, and is therefore determined by the conservation equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} = Q_S \tag{4.12}$$

where we call Q_S the salinity diffusion.

5 Derivation of the Primitive Equations

The goal of this chapter will be to derive the primitive equations (PEs) for the ocean and the atmosphere. We use the papers [1] and [2] as our main reference.

In the previous chapters, we derived equations describing atmospheric and oceanic fluids mathematically. Together we call them the *general equations of the ocean and the atmosphere.*

General equations of the ocean

To describe the ocean we need the momentum equation (4.11), the continuity equation (2.12), the thermodynamic equation (3.23), the equation of state (3.1) and the equation of diffusion of the salinity (4.12):

$$\rho \frac{\mathrm{d}\boldsymbol{V}_3}{\mathrm{d}t} + 2\rho\Omega \times \boldsymbol{V}_3 + \nabla p + \rho \boldsymbol{g} = D \tag{5.1}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \operatorname{div} \boldsymbol{V}_3 = 0 \tag{5.2}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = Q_T \tag{5.3}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = Q_S \tag{5.4}$$

$$\rho = f(T, S, p) \tag{5.5}$$

where $\boldsymbol{g}$ is the gravity vector, D the molecular dissipation, Q_T and Q_S the heat and salinity diffusions. It is easily seen that we have now 7 equations for 7 unknowns.

General equations of the atmosphere

For the atmosphere we can as well write down a set of general equations, containing the momentum equation (4.11), the continuity equation (2.12), the thermodynamic equation (3.14) and the equation of state (3.3), or in other words - 6 equations for 6 unknowns:

$$\frac{\mathrm{d}\boldsymbol{V}_3}{\mathrm{d}t} = -\frac{1}{\rho}\nabla p - \boldsymbol{g} - 2\Omega \times \boldsymbol{V}_3 + D \tag{5.6}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \operatorname{div} \boldsymbol{V}_3 = 0 \tag{5.7}$$

$$c_p \frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{5.8}$$

$$p = R\rho T \tag{5.9}$$

5.1 Spherical Coordinates

Since the earth can be seen as a sphere, it makes sense to consider our equations in spherical coordinates (θ, φ, r), where

$0 \leq \theta \leq \pi$ ... colatitude of the earth

$0 \leq \varphi \leq 2\pi$ ... longitude of the earth

r ... radical distance

a ... radius of the earth

$z = r - a$ ... vertical coordinate with respect to the sea level

as can be seen in Figure 5.1[1].

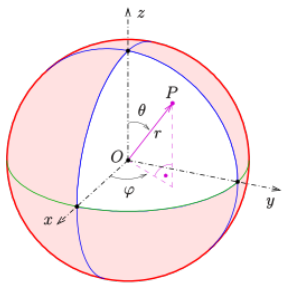

Figure 5.1: spherical coordinates

We know that the velocity of a fluid is the time derivative of a position. In terms of Cartesian coordinates, a position is described as

$$\mathbf{x} = \begin{pmatrix} r\cos\varphi\sin\theta \\ r\sin\varphi\sin\theta \\ r\cos\theta \end{pmatrix} \tag{5.10}$$

[1] *Attribution:* By Ag2gaeh (Own work) [CC BY-SA 4.0 (https://creativecommons.org/licenses/by-sa/4.0)], via Wikimedia Commons

The time derivative of this position can be found by

$$\dot{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial t} = \begin{pmatrix} \dot{r}\cos\varphi\sin\theta - r\sin\varphi\sin\theta\dot{\varphi} + r\cos\varphi\cos\theta\dot{\theta} \\ \dot{r}\sin\varphi\sin\theta + r\cos\varphi\sin\theta\dot{\varphi} + r\sin\varphi\cos\theta\dot{\theta} \\ \dot{r}\cos\theta - r\sin\theta\dot{\theta} \end{pmatrix} \tag{5.11}$$

$$= \dot{r}\underbrace{\begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}}_{e_z} + r\sin\theta\dot{\varphi}\underbrace{\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}}_{e_\varphi} + r\dot{\theta}\underbrace{\begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ -\sin\theta \end{pmatrix}}_{e_\theta} \tag{5.12}$$

where the dot always indicates the derivation with respect to time. See e.g. [9].

Velocity field

In this setting, let e_θ, e_φ, e_z be the unit-vectors in θ-, φ- and z-directions respectively. Knowing this, we can write the velocity field $\boldsymbol{V}_3$ as the total of the horizontal velocity field $v = v_\theta e_\theta + v_\varphi e_\varphi$ and the vertical velocity field $w = v_r e_z$:

$$\begin{aligned} \boldsymbol{V}_3 &= v + w \\ &= v_\theta e_\theta + v_\varphi e_\varphi + v_r e_z \qquad (5.13) \\ &= r\dot{\theta}e_\theta + r\sin\theta\dot{\varphi}e_\varphi + \dot{r}e_z \qquad (5.14) \end{aligned}$$

Material Derivative

The material derivative in the spherical coordinate setting yields

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + \frac{v_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} + v_z\frac{\partial}{\partial z} \tag{5.15}$$

which we can see after some calculation.

Coriolis Force

The Coriolis force needs to be written in terms of the according unit vectors as can be seen in [5]:

$$\Omega = \begin{pmatrix} 0 & 0 & -1 \\ -\sin\theta & \cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \Omega \\ 0 \end{pmatrix} = (0, \Omega\cos\theta, \Omega\sin\theta) \tag{5.16}$$

Hence:

$$2\Omega \times \boldsymbol{V}_3 = \begin{vmatrix} e_r & e_\theta & e_\varphi \\ 0 & 2\Omega\cos\theta & 2\Omega\sin\theta \\ v_r & v_\theta & v_\varphi \end{vmatrix} \tag{5.17}$$

$$= e_z(2\Omega v_\varphi\cos\theta - 2\Omega v_\theta\sin\theta) + e_\theta 2\Omega v_r\sin\theta - e_\varphi 2\Omega v_r\cos\theta \tag{5.18}$$

5.2 PEs of the Ocean

From theoretical and computational point of view, the general set of equations we found for the ocean and the atmosphere are too complicated to study. Therefore, we want to simplify this set of equations as much as possible, starting with the so-called *Boussinesq approximation*, see e.g. [1]

5.2.1 Boussinesq Approximation

In the Boussinesq approximation, we neglect density differences in the general equations, with exception of the equation of state and the buoyancy term in the momentum equation. Therefore, the Navier Stokes equations (5.1) and (5.2) simplify to

$$\rho_0 \frac{\mathrm{d}\boldsymbol{V}_3}{\mathrm{d}t} + 2\rho_0 \Omega \times \boldsymbol{V}_3 + \nabla p + \rho \boldsymbol{g} = D \tag{5.19}$$

$$\operatorname{div} \boldsymbol{V}_3 = 0 \tag{5.20}$$

where ρ_0 is some reference density. From the small depth assumption (the fact that the depth of the ocean is small compared to the radius of the earth a) we get the argument to further simplify the equations by replacing r by a to first order. Particularly, the material derivative (2.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{v_\theta}{a}\frac{\partial}{\partial \theta} + \frac{v_\varphi}{a \sin\theta}\frac{\partial}{\partial \varphi} + v_z \frac{\partial}{\partial z} \tag{5.21}$$

Taking into account the viscosity, our set of equations can be reformulated to represent the *Boussinesq equations of the ocean (BEs)* in spherical coordinates, see e.g. [1]:

$$\frac{\partial v}{\partial t} + \nabla_v v + w\frac{\partial v}{\partial z} + \frac{1}{\rho_0}\operatorname{grad} p + 2\Omega\cos\theta k \times v - \mu\Delta v - \nu\frac{\partial^2 v}{\partial z^2} = 0 \tag{5.22}$$

$$\frac{\partial w}{\partial t} + \nabla_v w + w\frac{\partial w}{\partial z} + \frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\rho}{\rho_0} g - \mu\Delta w - \nu\frac{\partial^2 w}{\partial z^2} = 0 \tag{5.23}$$

$$\operatorname{div} v + \frac{\partial w}{\partial z} = 0 \tag{5.24}$$

$$\frac{\partial T}{\partial t} + \nabla_v T + w\frac{\partial T}{\partial z} - \mu_T \Delta T - \nu_T \frac{\partial^2 T}{\partial z^2} = 0 \tag{5.25}$$

$$\frac{\partial S}{\partial t} + \nabla_v S + w\frac{\partial S}{\partial z} - \mu_S \Delta S - \nu_S \frac{\partial^2 S}{\partial z^2} = 0 \tag{5.26}$$

$$\rho = \rho_0(1 - \beta_T(T - T_0) + \beta_S(S - S_0)) \tag{5.27}$$

where the horizontal gradient operator and the horizontal divergence operator are defined as follows:

$$\operatorname{grad} p = \frac{1}{a}\frac{\partial p}{\partial \theta} e_\theta + \frac{1}{a \sin\theta}\frac{\partial p}{\partial \varphi} e_\varphi \tag{5.28}$$

$$\operatorname{div}(v_\theta e_\theta + v_\varphi e_\varphi) = \frac{1}{a\sin\theta}\left(\frac{\partial(v_\theta \sin\theta)}{\partial\theta} + \frac{\partial v_\varphi}{\partial\varphi}\right) \tag{5.29}$$

Also, as can be seen in [1], we can formulate the derivatives $\nabla_v \tilde{v}$ for a vector function $\tilde{v}$ and $\nabla_v \tilde{T}$ for a scalar function $\tilde{T}$ in spherical coordinates as

$$\nabla_v \tilde{T} = \frac{v_\theta}{a}\frac{\partial \tilde{T}}{\partial\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\partial \tilde{T}}{\partial\varphi} \tag{5.30}$$

$$\nabla_v \tilde{v} = \left\{\frac{v_\theta}{a}\frac{\partial \tilde{v}_\theta}{\partial\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\partial \tilde{v}_\theta}{\partial\varphi} - \frac{v_\varphi \tilde{v}_\varphi}{a}\cot\theta\right\} e_\theta, \tag{5.31}$$

$$+ \left\{\frac{v_\varphi}{a}\frac{\partial \tilde{v}_\varphi}{\partial\theta} + \frac{v_\varphi}{a\sin\theta}\frac{\partial \tilde{v}_\varphi}{\partial\varphi} + \frac{\tilde{v}_\theta v_\varphi}{a}\cot\theta\right\} e_\varphi, \tag{5.32}$$

and similarly, the Laplace-operators Δ for scalar functions and vector fields become

$$\Delta T = \frac{1}{a^2 \sin\theta}\left\{\frac{\partial}{\partial\theta}(\sin\theta \frac{\partial T}{\partial\theta}) + \frac{1}{\sin\theta}\frac{\partial^2 T}{\partial\varphi^2}\right\}, \tag{5.33}$$

$$\Delta v = \Delta(v_\theta e_\theta + v_\varphi e_\varphi) \tag{5.34}$$

$$= \left\{\Delta v_\theta - \frac{2\cos\theta}{a^2\sin^2\theta}\frac{\partial v_\varphi}{\partial\varphi} - \frac{v_\theta}{a^2\sin^2\theta}\right\} e_\theta \tag{5.35}$$

$$+ \left\{\Delta v_\varphi + \frac{2\cos\theta}{a^2\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi} - \frac{v_\varphi}{a^2\sin^2\theta}\right\} e_\varphi. \tag{5.36}$$

We also write the momentum equation (5.1) corresponding to the unit vectors of the velocity field, resulting in a set of two equations describing the momentum in every direction.

The Boussinesq equations are fully non-linear and 3-dimensional. Therefore, computational difficulties will occur. However, with this set of equations, it is already possible to get a mathematical analysis. In the next subsection, we will introduce the hydrostatic assumption, which is considered to be extremely accurate, and allows us to further simplify our set of equations, resulting in the *primitive equations of the large scale ocean.*

5.2.2 Hydrostatic Approximation

It is known that for the large scale ocean, the vertical scale is much smaller than the horizontal one. The scale analysis shows that the large-scale ocean satisfies the *hydrostatic approximation*:

$$\frac{\partial p}{\partial z} = -\rho \boldsymbol{g} \tag{5.37}$$

This equation connects the pressure and the density. Due to its high accuracy, this has become a fundamental equation in oceanography.

Our next step will be to replace the vertical momentum equation by the hydrostatic approximation. Doing this, we drop the vertical velocity, which is not explicitly given in the hydrostatic equation. Mathematical justification is given in [11], and will not be further discussed in this thesis.

As a result, we are not able to make predictions for w by doing so. We therefore have to find w through other means, which can cause some mathematical difficulties. Solutions to overcome these difficulties are given in [1].

The set of equations resulting from those approximations are called the *primitive equations of the large-scale ocean*:

$$\frac{\partial v}{\partial t} + \nabla_v v + w\frac{\partial v}{\partial z} + \frac{1}{\rho_0}\operatorname{grad} p + 2\Omega\cos\theta k \times v - \mu\Delta v - \nu\frac{\partial^2 v}{\partial z^2} = 0 \tag{5.38}$$

$$\frac{\partial p}{\partial z} = -\rho \boldsymbol{g} \tag{5.39}$$

$$\operatorname{div} v + \frac{\partial w}{\partial z} = 0 \tag{5.40}$$

$$\frac{\partial T}{\partial t} + \nabla_v T + w\frac{\partial T}{\partial z} - \mu_T\Delta T - \nu_T\frac{\partial^2 T}{\partial z^2} = 0 \tag{5.41}$$

$$\frac{\partial S}{\partial t} + \nabla_v S + w\frac{\partial S}{\partial z} - \mu_S\Delta S - \nu_S\frac{\partial^2 S}{\partial z^2} = 0 \tag{5.42}$$

$$\rho = \rho_0(1 - \beta_T(T - T_0) + \beta_S(S - S_0)) \tag{5.43}$$

The hydrostatic equations make the PEs suitable for numerical computations, which makes them fundamental for oceanographic simulations. Other models of the ocean can be derived from the PEs.

5.3 PEs of the Atmosphere

We get the primitive equations for the atmosphere the same way we got the primitive equations for the ocean. As before, we will work with a spherical coordinate system.

Consequently, we can rewrite the general equations of the atmosphere as

$$\frac{\mathrm{d}v_\theta}{\mathrm{d}t} + \frac{1}{r}(v_r v_\theta - v_\varphi^2 \cot\theta) = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + 2\Omega\cos\theta v_\varphi + D_\theta \tag{5.44}$$

$$\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} + \frac{1}{r}(v_r v_\varphi + v_\theta v_\varphi \cot\theta) = -\frac{1}{\rho r \sin\theta}\frac{\partial p}{\partial \varphi} - 2\Omega\cos\theta v_\theta - 2\Omega\sin\theta v_r + D_\varphi \tag{5.45}$$

$$\frac{\mathrm{d}v_r}{\mathrm{d}t} + \frac{1}{r}(v_\theta^2 - v_\varphi^2) = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \boldsymbol{g} + 2\Omega\sin\theta v_\varphi + D_r \tag{5.46}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta \sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r}\right) = 0 \tag{5.47}$$

$$c_p\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{5.48}$$

$$p = R\rho T \tag{5.49}$$

Again, we will neglect density differences as required for the Boussinesq approximation. Using this, we can rewrite the set of equations governing the atmosphere as

$$\frac{\mathrm{d}v_\theta}{\mathrm{d}t} - \frac{\partial v_\varphi^2}{\partial a}\cot\theta = -\frac{1}{\rho a}\frac{\partial p}{\partial\theta} + 2\Omega\cos\theta v_\varphi + D_\theta \tag{5.50}$$

$$\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} - \frac{\partial v_\theta v_\varphi^2}{\partial a}\cot\theta = -\frac{1}{\rho a\sin\theta}\frac{\partial p}{\partial\varphi} + 2\Omega\cos\varphi v_\theta + D_\varphi \tag{5.51}$$

$$\frac{\mathrm{d}v_r}{\mathrm{d}t} - \frac{1}{r}(v_\theta^2 + v_\varphi^2) = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \boldsymbol{g} + 2\Omega\sin\theta v_\varphi + D_r \tag{5.52}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\frac{1}{a\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta} + \frac{1}{a\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_r}{\partial r}\right) = 0 \tag{5.53}$$

$$c_p\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{5.54}$$

$$p = R\rho T \tag{5.55}$$

Next, we use the hydrostatic approximation (5.37) connecting the pressure and the density. Using all these approximations, we get the *primitive equations for the* *large-scale atmosphere*

$$\frac{\mathrm{d}v_\theta}{\mathrm{d}t} - \frac{\partial v_\varphi^2}{\partial a}\cot\theta = -\frac{1}{\rho a}\frac{\partial p}{\partial \theta} + 2\Omega\cos\theta v_\varphi + D_\theta \tag{5.56}$$

$$\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} - \frac{\partial v_\theta v_\varphi^2}{\partial a}\cot\theta = -\frac{1}{\rho a\sin\theta}\frac{\partial p}{\partial \varphi} + 2\Omega\cos\varphi v_\theta + D_\varphi \tag{5.57}$$

$$\frac{\partial p}{\partial r} = -\rho\boldsymbol{g} \tag{5.58}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\frac{1}{a\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta} + \frac{1}{a\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{\partial v_r}{\partial r}\right) = 0 \tag{5.59}$$

$$c_p\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{5.60}$$

$$p = R\rho T \tag{5.61}$$

5.4 Non-dimensional form of the PEs

5.4.1 Ocean

Let U, T_0, S_0, ρ_0 be reference values of the horizontal velocity, temperature, salinity and density respectively. Let a be the reference value for the horizontal length and Z the reference value of the vertical length. Then we set

$$\begin{array}{lll} v = v'U & w = w'\varepsilon U & T = T_0T' \\ S = S_0S' & \rho = \rho_0\rho' & p = \rho_0U^2p' \end{array} \tag{5.62}$$

$$t = \frac{a}{U}t' \qquad h = \frac{H}{Z} \qquad z = Zz' = \varepsilon az' \tag{5.63}$$

$$\begin{array}{lll} f = 2\cos\theta & \bar\beta_T = \beta_T T_0 & \bar\beta_S = \beta_S S_0 \\ \dfrac{1}{\mathrm{Re}_1} = \dfrac{\mu}{aU} & \dfrac{1}{\mathrm{Rt}_1} = \dfrac{\mu_T}{aU} & \dfrac{1}{\mathrm{Rs}_1} = \dfrac{\mu_S}{aU} \\ \dfrac{1}{\mathrm{Re}_2} = \dfrac{a\nu}{Z^2U} & \dfrac{1}{\mathrm{Rt}_2} = \dfrac{a\nu_T}{Z^2U} & \dfrac{1}{\mathrm{Rs}_2} = \dfrac{a\nu_S}{Z^2U} \end{array} \tag{5.64}$$

$$\begin{array}{llll} \bar b = \dfrac{\boldsymbol{g}Z}{U^2} & \mathrm{Ro} = \dfrac{U}{a\Omega} & \bar\alpha_T = Z\alpha_T & \bar\tau_v = \dfrac{Z\tau_v}{U} \\ \bar T_b = \dfrac{T_b}{T_0} & \bar T_A = \dfrac{T_A}{T_0} & \bar S_b = \dfrac{S_b}{S_0} & \varepsilon = \dfrac{Z}{a} \end{array} \tag{5.65}$$

where Re_i are the Reynolds numbers, Rt_i and Rs_i are the non-dimensional eddy-diffusion coefficients, Ro the Rossby number, which indicates how strongly the rotation of the earth influences the dynamical behaviour of the ocean.

From there, we can compute the non-dimensional form of the primitive equations of the ocean, resulting in:

$$\frac{\partial v}{\partial t} + \nabla v + w\frac{\partial v}{\partial z} + \operatorname{grad} p + \frac{1}{\mathrm{Ro}} f k \times v - \frac{1}{\mathrm{Re}_1}\Delta v - \frac{1}{\mathrm{Re}_2}\frac{\partial^2 v}{\partial z^2} = 0 \tag{5.66}$$

$$\frac{\partial p}{\partial z} + \bar{b}\rho = 0 \tag{5.67}$$

$$\operatorname{div} v + \frac{\partial w}{\partial z} = 0 \tag{5.68}$$

$$\frac{\partial T}{\partial t} + \nabla T + w\frac{\partial T}{\partial z} - \frac{1}{\mathrm{Rt}_1}\Delta T - \frac{1}{\mathrm{Rt}_2}\frac{\partial^2 T}{\partial z^2} = 0 \tag{5.69}$$

$$\frac{\partial S}{\partial t} + \nabla S + w\frac{\partial S}{\partial z} - \frac{1}{\mathrm{Rs}_1}\Delta S - \frac{1}{\mathrm{Rs}_2}\frac{\partial^2 S}{\partial z^2} = 0 \tag{5.70}$$

$$\rho = 1 - \bar{\beta}_t(T-1) + \bar{\beta}_S(S-1) \tag{5.71}$$

Remark 5.1. *We simplified the equations by dropping the super index prime.*

5.4.2 Atmosphere

Let, as before

$$\begin{aligned} v &= v'U \\ w &= \frac{P-p_0}{a}Uw' \\ T &= \bar{T}_0 T' \\ \Phi &= U^2\Phi' \end{aligned} \tag{5.72}$$

$$\begin{aligned} t &= \frac{a}{U}t' \\ p &= (P-p_0)\zeta + p_0 \\ f' &= 2\cos\theta \end{aligned} \tag{5.73}$$

$$\begin{aligned} &\frac{1}{\mathrm{Re}_1} = \frac{\mu_1}{aU} \qquad \frac{1}{\mathrm{Re}_2} = \frac{\nu_1 a \boldsymbol{g}^2}{UR^2\bar{T}_0^2}\left(\frac{P}{P-p_0}\right)^2 \\ &\frac{1}{\mathrm{Rt}_1} = \frac{\mu_2\bar{T}_0^2}{aU^3} \qquad \frac{1}{\mathrm{Rt}_2} = \frac{\nu_2 a\boldsymbol{g}^2}{U^3R^2}\left(\frac{P}{P-p_0}\right)^2 \\ &\mathrm{Ro} = \frac{U}{a\Omega} \\ &a_1 = \frac{R^2\bar{T}_0^2}{C^2U^2} \qquad b = \frac{R\bar{T}_0(P-p_0)}{U^2P} \\ &\bar{\alpha}_S = (P-p_0)\alpha_S \qquad \bar{T}_S = \frac{T_S}{\bar{T}_0} \end{aligned} \tag{5.74}$$

As above, we have the same parameters for the Reynolds numbers and Rossby number, which in this case measures the influence of the Earth's rotation on the behaviour of the atmosphere.

Using these relations, we get the primitive equations for the ocean in non-dimensional form (again, we omit the primes):

$$\frac{\partial v}{\partial t} + \nabla v + w\frac{\partial v}{\partial \zeta} + \frac{f}{\mathrm{Ro}} k \times v + \operatorname{grad}\Phi - \frac{1}{\mathrm{Re}_1}\Delta v - \frac{1}{\mathrm{Re}_2}\frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial v}{\partial \zeta}\right] = f_1 \tag{5.75}$$

$$\operatorname{div} v + \frac{\partial w}{\partial \zeta} = 0 \tag{5.76}$$

$$\frac{\partial \Phi}{\partial \zeta} + \frac{bP}{p}T = 0 \tag{5.77}$$

$$a_1\left(\frac{\partial T}{\partial t} + \nabla T + w\frac{\partial T}{\partial \zeta}\right) - \frac{bP}{p}w - \frac{1}{\mathrm{Rt}_1}\Delta T - \frac{1}{\mathrm{Rt}_2}\frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial T}{\partial \zeta}\right] = f_2 \tag{5.78}$$

6 | Special Settings

6.1 PEs of the atmosphere in the p-coordinate system

From the hydrostatic approximation (5.37) it is easily seen that p is decreasing with respect to the independent variable z, therefore we can perform a coordinate transformation of the form

$$(t, \theta, \varphi, z) \longrightarrow (t^*, \theta^*, \varphi^*, p = p(t, \theta, \varphi, z)) \tag{6.1}$$

While $z \in [0, \infty)$, it holds that $t \in [p_s, 0]$, namely from the pressure at sea or earth level to the pressure in the high atmosphere. We call this new system of variables the *pressure coordinate system* or *p-coordinate system.*

If we derive the primitive equations for the ocean from here with respect to the new coordinates, we obtain after some work, see e.g [1].

$$\frac{\mathrm{d}v_\theta}{\mathrm{d}t} - \frac{v_\varphi^2}{a}\cot\theta = -\frac{1}{a}\frac{\partial\Phi}{\partial\theta} + 2\Omega\cos\theta v_\varphi + D_\theta, \tag{6.2}$$

$$\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} - \frac{v_\theta v_\varphi}{a}\cot\theta = -\frac{1}{a\sin\theta}\frac{\partial\Phi}{\partial\varphi} + 2\Omega\cos\theta v_\theta + D_\varphi, \tag{6.3}$$

$$\frac{\partial\Phi}{\partial p} + \frac{R}{p}T = 0, \tag{6.4}$$

$$\frac{\partial w}{\partial p} + \frac{1}{a\sin\theta}\left(\frac{\partial v_\theta\sin\theta}{\partial\theta} + \frac{\partial v_\varphi}{\partial\varphi}\right) = 0, \tag{6.5}$$

$$c_p = \frac{\mathrm{d}T}{\mathrm{d}t} - \frac{RT}{p}w = Q_T, \tag{6.6}$$

where $\Phi = gz$ is the geopotential.

We can observe that the continuity equation in the p-coordinate system takes the same form as for an incompressible fluid. It is known that the atmosphere is a compressible fluid. Therefore, the form of the continuity equation is an advantage of the p-coordinate system.

6.2 Hydrostatic approximation with vertical viscosity

6.2.1 Ocean

When studying the long-term behaviour of the ocean, viscosity will play an important role in the dynamics. A common method is therefore to replace the hydrostatic approximation (5.37) by the following equation:

$$\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\rho}{\rho_0} g - \mu \Delta w - \nu \frac{\partial^2 w}{\partial z^2} = 0, \tag{6.7}$$

which we will call the *hydrostatic approximation with vertical viscosity.*

By doing this, we get the *primitive equations with vertical viscosity (PEV^2s)*:

$$\frac{\partial v}{\partial t} + \nabla_v v + w\frac{\partial v}{\partial z} + \frac{1}{\rho_0}\operatorname{grad} p + 2\Omega\cos\theta k \times v - \mu\Delta v - \nu\frac{\partial^2 v}{\partial z^2} = 0 \tag{6.8}$$

$$\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\rho}{\rho_0} g - \mu \Delta w - \nu \frac{\partial^2 w}{\partial z^2} = 0 \tag{6.9}$$

$$\operatorname{div} v + \frac{\partial w}{\partial z} = 0 \tag{6.10}$$

$$\frac{\partial T}{\partial t} + \nabla_v T + w\frac{\partial T}{\partial z} - \mu_T\Delta T - \nu_T\frac{\partial^2 T}{\partial z^2} = 0 \tag{6.11}$$

$$\frac{\partial S}{\partial t} + \nabla_v S + w\frac{\partial S}{\partial z} - \mu_S\Delta S - \nu_S\frac{\partial^2 S}{\partial z^2} = 0 \tag{6.12}$$

$$\rho = \rho_0(1 - \beta_T(T - T_0) + \beta_S(S - S_0)) \tag{6.13}$$

6.2.2 Atmosphere

Similarly, but with some computational effort, we can derive the PEV^2s with vertical viscosity for the atmosphere. See e.g. [2] for reference.

$$\frac{\partial v}{\partial t} + \nabla_v v + w\frac{\partial v}{\partial \zeta} + \frac{f}{\mathrm{Ro}} k \times v + \operatorname{grad}\Phi - \frac{1}{\mathrm{Re}_1}\Delta v - \frac{1}{\mathrm{Re}_2}\frac{\partial}{\partial \zeta} \\ - \frac{1}{\mathrm{Re}_2}\frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial v}{\partial \zeta}\right] = f_1 \tag{6.14}$$

$$\frac{\partial \Phi}{\partial \zeta} + \frac{bP}{p}T - \frac{\varepsilon^2}{\mathrm{Re}_1}\Delta w - \frac{\varepsilon^2}{\mathrm{Re}_2}\frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial w}{\partial \zeta}\right] = 0 \tag{6.15}$$

$$\operatorname{div} v + \frac{\partial w}{\partial \zeta} = 0 \tag{6.16}$$

$$a_1\left(\frac{\partial T}{\partial t} + \nabla_v T + w\frac{\partial T}{\partial \zeta}\right) - \frac{bP}{p}w - \frac{1}{\mathrm{Rt}_1}\Delta T - \frac{1}{\mathrm{Rt}_2}\frac{\partial}{\partial \zeta}\left[\left(\frac{p\bar{T}_0}{P\bar{T}}\right)^2 \frac{\partial T}{\partial \zeta}\right] = f_2 \tag{6.17}$$

7 | Boundary Conditions for the PEs

For a complete system of partial differential equations it is of course necessary to include boundary and initial conditions. In the following section we formulate suitable boundary conditions.
As for the initial conditions, they are necessary due to the derivations for time t in our equations. Therefore we choose appropriate values for temperature, salinity, density, pressure and velocity in our model.

7.1 Ocean

Since we decided to approximate r by a, we can also approximate the domain filled by the sea by $M_a \subset S_a^2 \times \mathbb{R}$:

$$M_a = \bigcup_{(\theta,\varphi)\in M_{a_h}} \{(\theta,\varphi)\} \times (-H(\theta,\varphi),0) \tag{7.1}$$

with S_a^2 the sphere of radius a, $M_{ah} \subset S_a^2$ a 2D-domain on the surface of the earth, which is occupied by the ocean, $H(\theta,\varphi)$ the depth of the ocean at the point of colatitude (θ,φ)

When imagining the ocean, we see a large area of water with some islands and continents in it. When we remove those pieces of land, defined as I_a^i, $(i = 1, 2, ..., N)$ from S_a^2 and only leave the water, we obtain the region M_{ah}:

$$M_{ah} = S_a^2 \setminus \left(\bigcup_{i=1}^{N} \bar{I}_a^i\right)$$

with the islands as simply connected open sets in S_a^2 with smooth boundary such that $\bar{I}_a^i \cap \bar{I}_a^j = \emptyset$. for $i \neq j$

The depth function $H : \overline{M}_{ah} \to R$ is a smooth and positive function with $H > 0$ in $\overline{M}_{ah}$, such that H is bounded from below on $\bar{M}_{ah}$: $H \geq H_o$ in $\overline{M}_{ah}$.

With these assumptions and domains, we are able to give the boundary values for our equation system:

$$\left.\begin{array}{l} \frac{\partial v}{\partial z} = \tau_v \\ w = 0 \\ \frac{\partial T}{\partial z} = \alpha_T(T_A - T) \\ \frac{\partial S}{\partial z} = 0 \end{array}\right\} \Gamma_u \qquad \text{upper surface of the ocean, } z = 0$$

$$\left.\begin{array}{l} (v, w) = 0 \\ (T, S) = (T_H, S_H) \end{array}\right\} \Gamma_b \qquad \text{bottom of the ocean, } z = -H$$

$$\left.\begin{array}{l} (v, w) = 0 \\ \frac{\partial}{\partial n}(T, S) = 0 \end{array}\right\} \Gamma_l \qquad \text{lateral boundary, } \bigcup_{(\theta,\varphi)\in M_{a_h}} \{(\theta, \varphi)\} \times (H(\theta, \varphi), 0)$$

where τ_v is the wind stress, α_T a positive constant related to the turbulent heating on the ocean surface, T_A the atmospheric equilibrium temperature, T_H and S_H the temperature and salinity at the bottom of the ocean.

7.2 Atmosphere

On the original coordinate system, we have no boundary conditions except some obvious conditions resulting from periodicy. There are, however, boundary conditions on the PEs if we write them in a p-coordinate-system, as can be seen in [2]

As before, we set p to be in the interval $[p_0, P]$, where $p_0 > 0$ is small and P approximates the pressure at the Earth surface. In other words, p_0 corresponds to the upper atmosphere and P to the surface of the Earth. The case where $p_0 = 0$ is a bit complicated and requires special attention.

The boundary conditions for the primitive equations of the atmosphere in a p-coordinate system are

$$p = P : \qquad (v, w) = 0 \qquad \frac{\partial T}{\partial p} = \alpha_S(T_S - T) \tag{7.2}$$

$$p = p_0 : \qquad (v, w) = 0 \qquad \frac{\partial T}{\partial p} = 0 \tag{7.3}$$

where $\alpha_S =$ const and relates to the the turbulent transition on the Earth's surface, and T_S is the given surface temperature.

For further insight, see [2].

8 Conclusion

We have started from deriving and collecting the general physical equations of a compressible fluid (under Coriolis force), to get a set of equations describing the ocean and the atmosphere. Under the aspect of physical effects, we were able to find an even better description for their behaviour.

However, since the general equations of the ocean and the atmosphere are not very efficient for mathematical purposes, we had to add further approximations in order to simplify our model. With the help of the Boussinesq approximation and the hydrostatic approximation we were able to simplify our set of equations and find the primitive equations of the ocean and the atmosphere, which are a very good mathematical approximation and are used to describe the behaviour of the two fluids.

On further discussion, we created a non-dimensional form for the primitive equations, and also took into account boundary conditions. For the atmosphere, it was furthermore possible to transform the coordinates to a p-coordinate system and get a continuity equation in the same form as for a incompressible fluid.

Another thing we looked into were the primitive equations with vertical viscosity, which are mainly used for discussions of the long-term behaviour of the ocean and the atmosphere.

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Linz, Oktober 2017

Michaela Lehner

Curriculum Vitae

Name: Michaela Lehner

Nationality: Austria

Date of Birth: 12 December, 1993

Place of Birth: Linz, Austria

Education:

1999–2003	Volksschule (elementary school) Ansfelden
2003–2012	Bundesrealgymnasium (secondary comprehensive school) Linz, Hamerlingstraße
2012–2017	Studies in Technical Mathematics, Johannes Kepler University Linz