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Institute of Computational Mathematics

A Robust Preconditioner for Distributed Optimal Control for Stokes Flow with Control Constraints

Markus Kollmann
Doctoral Program "Computational Mathematics" (W1214)
Johannes Kepler University
Altenberger Str. 69, 4040 Linz, Austria

Walter Zulehner
Institute of Computational Mathematics, Johannes Kepler University
Altenberger Str. 69, 4040 Linz, Austria

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A Robust Preconditioner for Distributed Optimal Control for Stokes Flow with Control Constraints

Markus Kollmann and Walter Zulehner

Abstract This work is devoted to the construction and analysis of robust solution techniques for the distributed optimal control problem for the Stokes equations with inequality constraints on the control. There the first order system of necessary and sufficient optimality conditions is nonlinear. A primal-dual active set method is applied in order to linearize the system. In every step a linear saddle point system has to be solved. For this system, we analyze a block-diagonal preconditioner that is robust with respect to the discretization parameter as well as the active set.

1 Introduction

Velocity tracking plays an important role in fluid mechanics. There the main focus is to steer the velocity to a desired state (target velocity) by controlling it by some force, which is typically restricted by inequality constraints. The corresponding nonlinear optimality system can be solved by a primal-dual active set method, which is equivalent to a semi-smooth Newton method (cf. [8]). The resulting linear system in each Newton step is a parameter dependent saddle point problem. In this paper we discuss the preconditioned MinRes method for solving these linear problems robustly with respect to the discretization parameter and the involved active set.

A similar approach is presented in [7] for a distributed optimal control of elliptic equations with various types of inequality constraints, where a preconditioner is

Markus Kollmann
Doctoral Program Computational Mathematics, Johannes Kepler University, Altenbergerstr. 69, A-4040 Linz, Austria, e-mail: markus.kollmann@dk-compmath.jku.at

Walter Zulehner
Institute of Computational Mathematics, Johannes Kepler University, Altenbergerstr. 69, A-4040 Linz, Austria, e-mail: zulehner@numa.uni-linz.ac.at

constructed based on the mapping properties of the involved operators in Sobolev spaces equipped with the standard norms.

Our construction of a preconditioner and the analysis are based on a result in [14], where a block-diagonal preconditioner for the distributed optimal control problem for the Stokes equations without constraints is presented, which is robust with respect to the discretization parameter and the model parameter. This preconditioner is based on the mapping properties of the involved operators in Sobolev spaces equipped with non-standard norms. Here we use this preconditioner also in the case of constrained optimization and show its robustness with respect to the discretization parameter and the involved active set.

The paper is organized as follows: In Sect. 2 we introduce the velocity tracking problem, discretize it by a finite element method and derive the resulting linearized system. Section 3 deals with the analysis of our block-diagonal preconditioner used in a MinRes method for the linear system. In Sect. 4 we present some numerical experiments. The paper ends with a few concluding remarks.

2 The Optimal Control Problem

As a model problem, we consider the following velocity tracking problem for Stokes flow with distributed control: Find the velocity $\mathbf{u} \in H_0^1(\Omega)^d$, the pressure $p \in L_0^2(\Omega) := \left\{q \in L^2(\Omega) : \int_\Omega q\, dx = 0\right\}$, and the force $\mathbf{f} \in L^2(\Omega)^d$ that minimizes the cost functional

$$J(\mathbf{u},\mathbf{f}) = \frac{1}{2}||\mathbf{u}-\mathbf{u_d}||^2_{L^2(\Omega)} + \frac{\alpha}{2}||\mathbf{f}||^2_{L^2(\Omega)} , \qquad (1)$$

subject to the state equations

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega , \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega , \\ \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma , \quad \mathbf{f_a} \le \mathbf{f} \le \mathbf{f_b} \quad \text{a.e. in } \Omega . \end{aligned}$$

Here Ω is an open and bounded domain in $\mathbb{R}^d$ ($d \in \{1,2,3\}$) with Lipschitz-continuous boundary Γ, $\mathbf{u}_d \in L^2(\Omega)^d$ is the desired velocity, $\alpha > 0$ is a cost parameter and $\mathbf{f_a}$, $\mathbf{f_b} \in L^2(\Omega)^d$ are the lower and upper bounds for the control variable $\mathbf{f}$, respectively.

In order to solve this optimal control problem, we consider the first-discretize-then-optimize strategy. As an example of a discretization method we discuss the finite element method using the Taylor-Hood element on a simplicial subdivision of Ω consisting of continuous and piecewise quadratic functions for the velocity and the force and continuous and piecewise linear functions for the pressure.

The discrete counterpart of (1) is:

$$\text{Minimize} \quad \frac{1}{2}(\mathbf{u_h}-\mathbf{u_{d_h}})^T\mathbf{M}(\mathbf{u_h}-\mathbf{u_{d_h}}) + \frac{\alpha}{2}\mathbf{f_h}^T\mathbf{M}\mathbf{f_h} , \qquad (2)$$

subject to the state equations

$$\begin{aligned} \mathbf{K}\mathbf{u_h} - \mathbf{D}^T\mathbf{p_h} &= \mathbf{M}\mathbf{f_h}\ , \\ -\mathbf{D}\mathbf{u_h} &= \mathbf{0}\ , \\ \mathbf{f_{a_h}} \le \mathbf{f_h} &\le \mathbf{f_{b_h}}\ , \end{aligned}$$

where $\mathbf{M}$ denotes the mass matrix representing the $L^2(\Omega)^d$ scalar product, $\mathbf{K}$ denotes the stiffness matrix representing the vector Laplace operator on the finite element space, $\mathbf{D}$ denotes the divergence matrix representing the divergence operator on the involved finite element spaces and $\mathbf{u_h}$, $\mathbf{f_h}$ and $\mathbf{p_h}$ are the coordinate vectors of $\mathbf{u}$, $\mathbf{f}$ and p w.r.t. the nodal basis, respectively.

The first order system of necessary and sufficient optimality conditions of (2) can be expressed as follows (cf. [5] for the continuous case):

$$\left.\begin{aligned} \mathbf{M}\mathbf{u_h} + \mathbf{K}\hat{\mathbf{u}}_\mathbf{h} - \mathbf{D}^T\hat{\mathbf{p}}_\mathbf{h} &= \mathbf{M}\mathbf{u_{d_h}}\ , \\ -\mathbf{D}\hat{\mathbf{u}}_\mathbf{h} &= \mathbf{0}\ , \\ \alpha\mathbf{M}\mathbf{f_h} - \mathbf{M}\hat{\mathbf{u}}_\mathbf{h} + \mathbf{z_h} &= \mathbf{0}\ , \\ \mathbf{K}\mathbf{u_h} - \mathbf{D}^T\mathbf{p_h} - \mathbf{M}\mathbf{f_h} &= \mathbf{0}\ , \\ -\mathbf{D}\mathbf{u_h} &= \mathbf{0}\ , \\ \mathbf{z_h} - \max\left\{\mathbf{0}, \mathbf{z_h} + c(\mathbf{f_h} - \mathbf{f_{b_h}})\right\} - \min\{\mathbf{0}, \mathbf{z_h} - c(\mathbf{f_{a_h}} - \mathbf{f_h})\} &= \mathbf{0}\ , \end{aligned}\right\} \tag{3}$$

for any $c > 0$ with Lagrange multipliers $\hat{\mathbf{u}}_\mathbf{h}$, $\hat{\mathbf{p}}_\mathbf{h}$ and $\mathbf{z_h}$.

In order to solve this system, we propose a primal-dual active set method as introduced in [1]. The strategy proceeds as follows: Given the k-th iterate $(\mathbf{u_{h,k}}, \mathbf{p_{h,k}}, \mathbf{f_{h,k}}, \hat{\mathbf{u}}_\mathbf{h,k}, \hat{\mathbf{p}}_\mathbf{h,k}, \mathbf{z_{h,k}})$, the active sets are determined by

$$\begin{aligned} \mathscr{A}_k^+ &= \left\{i : z_{h,k}^i + c(f_{h,k}^i - f_{b_h}^i) > 0\right\}\ , \\ \mathscr{A}_k^- &= \left\{i : z_{h,k}^i - c(f_{a_h}^i - f_{h,k}^i) < 0\right\}\ , \end{aligned}$$

where $z_{h,k}^i$ is the i-th component of $\mathbf{z_{h,k}}$ and the inactive set $\mathscr{I}_k$ is the complement of $\mathscr{A}_k^+ \cup \mathscr{A}_k^-$ in the set of all indices. One step of the primal-dual active set method for the solution of (3), given in terms of the new iterate, reads as follows:

$$\begin{pmatrix} \mathbf{M} & \mathbf{0} & \mathbf{0} & \mathbf{K} & -\mathbf{D}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{D} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha\mathbf{M} & -\mathbf{M} & \mathbf{0} & \mathbf{I} \\ \mathbf{K} & -\mathbf{D}^T & -\mathbf{M} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{D} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & c\chi_{\mathscr{A}_\mathbf{k}} & \mathbf{0} & \mathbf{0} & \chi_{\mathscr{I}_\mathbf{k}} \end{pmatrix} \begin{pmatrix} \mathbf{u_{h,k+1}} \\ \mathbf{p_{h,k+1}} \\ \mathbf{f_{h,k+1}} \\ \hat{\mathbf{u}}_\mathbf{h,k+1} \\ \hat{\mathbf{p}}_\mathbf{h,k+1} \\ \mathbf{z_{h,k+1}} \end{pmatrix} = \begin{pmatrix} \mathbf{M}\mathbf{u_{d_h}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ c(\chi_{\mathscr{A}_\mathbf{k}^+}\mathbf{f_{b_h}} + \chi_{\mathscr{A}_\mathbf{k}^-}\mathbf{f_{a_h}}) \end{pmatrix}\ , \tag{4}$$

where $\chi_{\mathscr{A}_\mathbf{k}^+}$, $\chi_{\mathscr{A}_\mathbf{k}^-}$ and $\chi_{\mathscr{A}_\mathbf{k}}$ are diagonal 0-1-matrices representing the characteristic functions of $\mathscr{A}_k^+$, $\mathscr{A}_k^-$ and $\mathscr{A}_k = \mathscr{A}_k^+ \cup \mathscr{A}_k^-$, respectively. Since we focus here on the solution of individual steps, we drop the iteration index from now on.

By eliminating $\mathbf{f_h}$ and $\mathbf{z_h}$ from the third and the last line the resulting system $\mathscr{K}\, x = b$ reads:

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} w_h \\ \hat{w}_h \end{pmatrix} = \begin{pmatrix} e_h \\ g_h \end{pmatrix} , \tag{5}$$

with

$$A = \begin{pmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} , \quad B = \begin{pmatrix} \mathbf{K} & -\mathbf{D}^T \\ -\mathbf{D} & \mathbf{0} \end{pmatrix} = B^T , \quad C = \begin{pmatrix} \alpha^{-1}\mathbf{M}_{\mathbf{C}_\mathscr{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} ,$$
$$e_h = \begin{pmatrix} \mathbf{M}\mathbf{u}_{\mathbf{d_h}} \\ \mathbf{0} \end{pmatrix} , \quad g_h = \begin{pmatrix} \mathbf{0} \\ \mathbf{g}_{\mathbf{h_1}} \end{pmatrix} , \quad w_h = \begin{pmatrix} \mathbf{u_h} \\ \mathbf{p_h} \end{pmatrix} , \quad \hat{w}_h = \begin{pmatrix} \hat{\mathbf{u}}_\mathbf{h} \\ \hat{\mathbf{p}}_\mathbf{h} \end{pmatrix} ,$$

where

$$\mathbf{M}_{\mathbf{C}_\mathscr{A}} = \mathbf{M} - \mathbf{P}_\mathscr{A}{}^T \left(\mathbf{P}_\mathscr{A}\mathbf{M}^{-1}\mathbf{P}_\mathscr{A}{}^T\right)^{-1} \mathbf{P}_\mathscr{A} ,$$
$$\mathbf{g}_{\mathbf{h_1}} = \mathbf{P}_\mathscr{A}{}^T \left(\mathbf{P}_\mathscr{A}\mathbf{M}^{-1}\mathbf{P}_\mathscr{A}{}^T\right)^{-1} \left(\mathbf{P}_{\mathscr{A}^+}\mathbf{f}_{\mathbf{b_h}} + \mathbf{P}_{\mathscr{A}^-}\mathbf{f}_{\mathbf{a_h}}\right) ,$$

and $\mathbf{P}_\mathscr{A}$ is a rectangular matrix consisting of those rows of $\chi_\mathscr{A}$ which belong to the active indices, similar for $\mathbf{P}_{\mathscr{A}^\pm}$.

The system matrix $\mathscr{K}$ is symmetric and indefinite. For solving the corresponding linear system we propose a MinRes method, see, e.g., [13]. Without preconditioning the convergence rate would deteriorate with respect to the discretization parameter h and the cost parameter α. Therefore, preconditioning is an important issue.

3 A Robust Preconditioning Technique

In this section, we discuss a preconditioning strategy for the saddle point system (5). Due to the symmetry and coercivity properties of the underlying operators the blocks fulfill the following properties: $\mathbf{K} = \mathbf{K}^T > 0$, $\mathbf{M} = \mathbf{M}^T > 0$ and $\mathbf{M}_{\mathbf{C}_\mathscr{A}} = \mathbf{M}_{\mathbf{C}_\mathscr{A}}{}^T \geq 0$. For our choice of the finite element functions, $\mathbf{D}$ is of full rank.

In [14] a block-diagonal preconditioner is constructed for the distributed optimal control problem of the Stokes equations without constraints on the control. Its robustness w.r.t. h and α is shown for this case, which corresponds to the setting $\mathscr{A} = \emptyset$.

This preconditioner reads as follows:

$$\mathscr{P} = \text{ diag } (P_1, P_2) , \tag{6}$$

where $P_1 = \text{ diag } \left(\mathbf{P}, \alpha\mathbf{D}\mathbf{P}^{-1}\mathbf{D}^T\right)$ and $P_2 = \alpha^{-1}P_1$ with $\mathbf{P} = \mathbf{M} + \alpha^{1/2}\mathbf{K}$. The idea is now to use this preconditioner also in our case. The next theorem contains the main result of this paper, where we use the following notation: For any symmetric and positive (semi-) definite matrix Q we denote by $\|\cdot\|_Q$ the energy (semi-) norm induced by Q.

Theorem 1. *The system matrix $\mathscr{K}$ of (5) satisfies:*

$$\underline{c}\|x\|_{\mathscr{P}} \le \sup_{y\neq 0} \frac{y^T \mathscr{K} x}{\|y\|_{\mathscr{P}}} \le \overline{c}\|x\|_{\mathscr{P}} \quad \forall x ,$$

with constants $\underline{c}$, $\overline{c}$ independent of the discretization parameter h and the active set $\mathscr{A}$.

Proof. Due to Theorem 2.6 from [14], it is necessary and sufficient to prove

$$\underline{c}_1 \|w_h\|^2_{P_1} \le \|w_h\|^2_A + \sup_{\hat{r}_h\neq 0} \frac{\left(\hat{r}_h^T B w_h\right)^2}{\|\hat{r}_h\|^2_{P_2}} \le \overline{c}_1 \|w_h\|^2_{P_1} , \tag{7}$$

$$\underline{c}_2 \|\hat{w}_h\|^2_{P_2} \le \|\hat{w}_h\|^2_C + \sup_{r_h\neq 0} \frac{\left(r_h^T B \hat{w}_h\right)^2}{\|r_h\|^2_{P_1}} \le \overline{c}_2 \|\hat{w}_h\|^2_{P_2} , \tag{8}$$

with $r_h = \begin{pmatrix} \mathbf{v_h} \\ \mathbf{q_h} \end{pmatrix}$, $\hat{r}_h = \begin{pmatrix} \hat{\mathbf{v}}_\mathbf{h} \\ \hat{\mathbf{q}}_\mathbf{h} \end{pmatrix}$ for constants $\underline{c}_1$, $\overline{c}_1$, $\underline{c}_2$ and $\overline{c}_2$ independent of the discretization parameter h and the active set $\mathscr{A}$. For proving (7), we first show

$$\underline{c}_3 \|w_h\|_{P_1} \le \sup_{\hat{r}_h\neq 0} \frac{\hat{r}_h^T B w_h}{\|\hat{r}_h\|_{P_1}} \le \overline{c}_3 \|w_h\|_{P_1} , \tag{9}$$

for constants $\underline{c}_3$ and $\overline{c}_3$ independent of the discretization parameter h and the active set $\mathscr{A}$. In order to prove (9) we have to verify the conditions of the Theorem of Brezzi [3]:

The boundedness of the bilinear forms, say a and b, associated with $\mathbf{K}$ and $\mathbf{D}$ is trivial. Using Friedrichs inequality with constant c_F we can show the coercivity of a:

$$a(\mathbf{u_h},\mathbf{u_h}) = \|\mathbf{u_h}\|^2_\mathbf{K} \ge \frac{1}{2c_F}\|\mathbf{u_h}\|^2_\mathbf{M} + \frac{1}{2\sqrt{\alpha}}\sqrt{\alpha}\|\mathbf{u_h}\|^2_\mathbf{K} \ge \min\left\{\frac{1}{2c_F}, \frac{1}{2\sqrt{\alpha}}\right\}\|\mathbf{u_h}\|^2_\mathbf{P} .$$

Since

$$\sup_{\hat{\mathbf{v}}_\mathbf{h}\neq\mathbf{0}} \frac{b(\hat{\mathbf{v}}_\mathbf{h},\mathbf{p_h})}{\|\hat{\mathbf{v}}_\mathbf{h}\|_\mathbf{P}} = \sup_{\hat{\mathbf{v}}_\mathbf{h}\neq\mathbf{0}} \frac{\mathbf{p_h}^T\mathbf{D}\hat{\mathbf{v}}_\mathbf{h}}{\|\hat{\mathbf{v}}_\mathbf{h}\|_\mathbf{P}} = \|\mathbf{p_h}\|_{\mathbf{D}\mathbf{P}^{-1}\mathbf{D}^T} = \frac{1}{\sqrt{\alpha}}\|\mathbf{p_h}\|_{\alpha\mathbf{D}\mathbf{P}^{-1}\mathbf{D}^T} ,$$

the inf-sup condition of b is satisfied. Hence (9) follows.

From (9) and the fact that $P_2 = \alpha^{-1}P_1$ we get

$$\sqrt{\alpha}\underline{c}_3 \|w_h\|_{P_1} \le \sup_{\hat{r}_h\neq 0} \frac{\hat{r}_h^T B w_h}{\|\hat{r}_h\|_{P_2}} \le \sqrt{\alpha}\overline{c}_3 \|w_h\|_{P_1} . \tag{10}$$

Furthermore we have

$$0 \le \|w_h\|_A^2 = \|\mathbf{u_h}\|_{\mathbf{M}}^2 \le \|w_h\|_{P_1}^2 \ . \tag{11}$$

Therefore, combining (10) with (11) yields (7). Equation (8) can be shown analogously. $\square$

As a consequence of Theorem 1 we have:

$$\kappa(\mathscr{P}^{-1}\mathscr{K}) := \|\mathscr{P}^{-1}\mathscr{K}\|_{\mathscr{P}}\|\mathscr{K}^{-1}\mathscr{P}\|_{\mathscr{P}} \le \frac{\overline{c}}{\underline{c}} \ , \tag{12}$$

i.e., the condition number of the preconditioned system is bounded independently of h and $\mathscr{A}$. Therefore, the number of iterations of the preconditioned MinRes method can be bounded independently of h and $\mathscr{A}$ (see e.g. [6]).

Remark 1. The result of Theorem 1 can be shown not only on the discrete level but also on the continuous level using the corresponding non-standard norms in $H_0^1(\Omega) \times L_0^2(\Omega)$ for $\mathbf{u}$ and p as well as for the Lagrange multipliers $\hat{\mathbf{u}}$ and $\hat{p}$.

Remark 2. Using the standard norms in $H_0^1(\Omega) \times L_0^2(\Omega)$, as it is done in [7] for the elliptic case, leads to the preconditioner:

$$\mathscr{P}_s = \ \mathrm{diag}\ (\mathbf{K}, \mathbf{M_p}, \mathbf{K}, \mathbf{M_p}) \ , \tag{13}$$

where $\mathbf{M_p}$ denotes the mass matrix for the pressure element. In this case, one can show a similar result as in Theorem 1.

Remark 3. If we consider the distributed optimal control problem for the Stokes equations with different observation and control domains Ω_1 and Ω_2, we end up with the following linear system:

$$\begin{pmatrix} \mathbf{M_1} & \mathbf{0} & \mathbf{K} & -\mathbf{D}^T \\ \mathbf{0} & \mathbf{0} & -\mathbf{D} & \mathbf{0} \\ \mathbf{K} & -\mathbf{D}^T & -\alpha^{-1}\mathbf{M_2} & \mathbf{0} \\ -\mathbf{D} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u_h} \\ \mathbf{p_h} \\ \hat{\mathbf{u}}_\mathbf{h} \\ \hat{\mathbf{p}}_\mathbf{h} \end{pmatrix} = \begin{pmatrix} \mathbf{Mu_{d_h}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \ ,$$

where $\mathbf{M_1}$ and $\mathbf{M_2}$ are the mass matrices on Ω_1 and Ω_2, respectively. With the preconditioner $\mathscr{P}$ from above, one can show a similar result as in Theorem 1 with robustness w.r.t. h, Ω_1 and Ω_2.

4 Numerical Experiments

The numerical experiments are carried out on the unit square domain $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$. The initial mesh contains four triangles obtained by connecting the two diagonals. The final mesh was constructed by applying k uniform refinement steps to the initial mesh, leading to a meshsize $h = 2^{-k}$. For constructing a practically realizable preconditioner we proceed as follows: First we replace the matrix $\mathbf{D}\left(\mathbf{M} + \alpha^{1/2}\mathbf{K}\right)^{-1}\mathbf{D}^T$ by $\left(\alpha^{1/2}\mathbf{M_p}^{-1} + \mathbf{K_p}^{-1}\right)^{-1}$ as proposed in [4], where $\mathbf{K_p}$

denotes the stiffness matrix for the pressure element. Then the application of the preconditioner would require the multiplication of a vector from the left by the inverse of the matrices $\mathbf{M}+\alpha^{1/2}\mathbf{K}$, $\mathbf{M_p}$ and $\mathbf{K_p}$. These actions are replaced by one step of a V-cycle iteration for $\mathbf{M}+\alpha^{1/2}\mathbf{K}$ and $\mathbf{K_p}$ and by one step of a symmetric Gauss-Seidel iteration for $\mathbf{M_p}$. The V-cycle is done with one step of a symmetric Gauss-Seidel iteration for the pre-smoothing process and for the post-smoothing process. The resulting realizable preconditioner is spectrally equivalent to the theoretical preconditioner according to the analysis in [2, 12, 9, 11, 10].

We demonstrate the efficiency of our solver with two different prescribed active sets.

As a first test case, the active set $\mathscr{A}$ is chosen as the set of all indices of those nodes which lie in the upper half of the computational domain. Table 1 shows the condition number of the preconditioned system matrix with preconditioner $\mathscr{P}$ for various values of h and α, where k denotes the number of refinements, N is the total number of degrees of freedom of the discretized optimality system (5). In the second test case, the active set $\mathscr{A}$ is chosen as a randomly distributed set, having the same cardinality as in the first test case. Table 2 shows the condition number of the preconditioned system matrix with preconditioner $\mathscr{P}$.

Table 1 Condition numbers

k	N	α							
		10^{-7}	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1
4	9 030	>500	70.9	13.92	7.49	8.12	8.68	9.2	9.53
5	36 486	>500	74	14.16	8.25	8.81	9.3	9.72	9.99
6	146 694	>500	79	14.63	8.87	9.46	9.92	10.16	10.34
7	588 294	>500	83	15.21	9.06	9.79	10.25	10.47	10.66

Table 2 Condition numbers

k	N	α			
		10^{-12}	10^{-8}	10^{-4}	1
4	9 030	>500	7.34	7.41	9.52
5	36 486	>500	4.95	8.21	9.98
6	146 694	133	6.13	8.88	10.34
7	588 294	16.65	6.71	9.12	10.58

Additional numerical experiments using the preconditioner $\mathscr{P}_s$ showed that the preconditioner $\mathscr{P}$ has a better performance than the standard one, e.g., while the preconditioner $\mathscr{P}_s$ behaves reasonably only for $\alpha \geq 10^{-2}$, the preconditioner $\mathscr{P}$ behaves reasonably as long as $\alpha \geq 10^{-5}$.

5 Concluding Remarks

In order to develop a robust solver for the linear system (5) we used the block-diagonal preconditioning technique introduced in [14]. The preconditioner constructed there was reused for the control constrained distributed optimal control problem for the Stokes equations and robustness w.r.t. the discretization parameter as well as the active set was shown. Even though the preconditioner is not robust w.r.t. α, the numerical experiments show a good performance of this preconditioner as long as α is not extremely small.

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