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A Finite Element Solver for a Multiharmonic Parabolic Optimal Control Problem

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A FINITE ELEMENT SOLVER FOR A MULTIHARMONIC PARABOLIC OPTIMAL CONTROL PROBLEM

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Abstract. This paper presents the analysis of a distributed parabolic optimal control problem in a multiharmonic setting. In particular, the desired state is assumed to be multiharmonic. After eliminating the control from the optimality system, we arrive at the reduced optimality system for the state and the co-state that is nothing but a coupled system of a forward and a backward parabolic partial differential equation. Due to the linearity, the state and the co-state are multiharmonic as well. We discretize the Fourier coefficients by the finite element method. This leads to a large system of algebraic equations, which fortunately decouples into smaller systems each of them defining the cosine and sine Fourier coefficients for the state and co-state with respect to a single frequency. For these smaller systems we construct preconditioners resulting in a fast converging minimal residual solver with a parameter-independent convergence rate. All these systems can be solved totally in parallel.

1. Introduction

In this paper, we focus on the construction and analysis of efficient and robust solvers for multiharmonic parabolic optimal control problems where we want to minimize a linear functional of the form

$$\mathcal{J}(y,u) = \mathcal{F}(y) + \mathcal{G}(u). \tag{1}$$

under PDE (Partial Differential Equation) constraints. In our model problem, the state y is the solution of a second-order parabolic partial differential equation in a bounded three-dimensional Lipschitz domain. We want to reach a certain desired distribution y_d, which is described through the functional $\mathcal{F}(y)$, whereas the functional $\mathcal{G}(u)$ reflects the costs of the control u and provides a regularization as well. For further information and a general analysis of optimal control problems, we refer, for example, to Lions [25] and Tröltzsch [33]. In this paper, we consider the case of a distributed control, which acts on the whole space-time cylinder.

In many practical applications, especially, in electromagnetics, we can assume that the desired state y_d and, therefore, also the state y and the control u are time-periodic, see e.g. Gunzburger and Trenchea [14]. A very important tool for the treatment of time-periodic problems is the multiharmonic approach where the solution is approximated by a truncated Fourier series. Earlier works on the use of the multiharmonic ansatz for simulating electromagnetic devices in the frequency domain can be found e.g. in the papers by Yamada and Bessho [36], Paoli et al. [30], Gersem et al. [12], Gyselinck et al. [15] and Bachinger et. al [5]. In Bachinger et al. [6, 7], Copeland et al. [10, 11] and Kolmbauer and Langer [19], the multiharmonic approach has successfully been applied to linear and nonlinear eddy-current problems. Other applications are mentioned in [1] where nested multigrid methods for solving time-periodic parabolic optimal control problems are studied. This approach is based on the reduction of the first order optimality condition to an operator equation of second kind and its solution by means of a so-called multigrid method of second kind, as proposed by Hackbusch [16].

In this paper, we assume from the very beginning that the desired state y_d is multiharmonic. This yields a multiharmonic representation for the state y, the co-state p and the control u as well. Let us mention that this assumption is very reasonable in many practical applications. Moreover, in the general time-periodic case, the desired state y_d is first approximated by a truncated Fourier series that brings us back to the multiharmonic setting. In the case of linear optimal control problems, the advantage of the multiharmonic ansatz is that the optimality system, which depends only on the Fourier coefficients, decouples: each block is related to a single mode, and can be solved independently. The linear systems corresponding to these blocks are saddle point problems. The fast solution of these linear systems requires a good preconditioner. In our model problem, some critical model parameters are involved (see Section 5) as well as the discretization parameter from the finite element method. We choose the preconditioned Minimal Residual (MINRES) method for solving the linear system. A new technique for constructing preconditioners for saddle point problems was presented in the works of Schöberl and Zulehner [31] and Zulehner [38], which is based on interpolation theory. We follow the strategy of [38] and construct block-diagonal preconditioners for the MINRES solver of the linear system finally leading to parameter independent and hence fast convergence rates. We mention that the blocks of the preconditioner originally obtained by space interpolation or norm equivalence techniques have to be replaced by easily invertible blocks in order to obtain optimal or almost optimal complexity. These inexact versions of our original preconditioner can be derived by standard preconditioning techniques like multigrid, multilevel, or domain decomposition preconditioning methods.

The rest of the paper is organized as follows. In Section 2, we introduce the multiharmonic parabolic optimal control problem that is investigated in this paper. The corresponding optimality system is derived in Section 3. In Section 4, we present and analyze the multiharmonic finite element method that is used for the time-space discretization of the optimality system. The derivation and the analysis of a robust preconditioner for the discrete optimality system is the main new contribution of the paper. The new preconditioner together with the corresponding MINRES solver is presented in Section 5. Section 6 is devoted to the discretization error analysis. Finally, we discuss some numerical results in Section 7 and draw some conclusions in Section 8.

2. A Multiharmonic Parabolic Optimal Control Problem

Let us assume that the domain $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain with boundary $\Gamma := \partial\Omega$. For our distributed optimal control problem, the cost functional $\mathcal{J}(y,u) = \mathcal{F}(y) + \mathcal{G}(u)$ is given by the relations

$$\mathcal{F}(y) = \frac{1}{2}\int_0^T \int_\Omega [y(\boldsymbol{x},t) - y_d(\boldsymbol{x},t)]^2 \, d\boldsymbol{x}\, dt \tag{2}$$

and

$$\mathcal{G}(u) = \frac{\lambda}{2}\int_0^T \int_\Omega [u(\boldsymbol{x},t)]^2 \, d\boldsymbol{x}\, dt, \tag{3}$$

where y_d is some given multiharmonic desired state, and $\lambda > 0$ is some positive parameter providing a cost weighting of the control u and, at the same time, a regularization of the optimal control problem. In the following, we denote by $Q_T := \Omega \times (0,T)$ the space-time cylinder and by $\Sigma_T := \Gamma \times (0,T)$ its mantle boundary, where $T > 0$ is the time period. Our linear parabolic optimal control problem is now given by finding the minimum of the cost functional $\mathcal{J}$,

$$\min_{y,u} \mathcal{J}(y,u), \tag{4}$$

subject to the state equations

$$\begin{cases} \sigma(\boldsymbol{x})\dfrac{\partial}{\partial t}y(\boldsymbol{x},t) - \nabla\cdot(\nu(\boldsymbol{x})\nabla y(\boldsymbol{x},t)) = u(\boldsymbol{x},t) & \forall(\boldsymbol{x},t)\in Q_T, \\ y(\boldsymbol{x},t) = 0 & \forall(\boldsymbol{x},t)\in\Sigma_T, \\ y(\boldsymbol{x},0) = y(\boldsymbol{x},T) & \forall\boldsymbol{x}\in\Omega, \end{cases} \tag{5}$$

where the symbol ∇ denotes the nabla operator with respect to the space variables $\boldsymbol{x} = (x_1, x_2, x_3)$. For the time being, we assume that the coefficients $\sigma(\cdot)$ and $\nu(\cdot)$ are uniformly positive and bounded. In practical applications, e.g. in computational electromagnetics, $\sigma(\cdot)$ and $\nu(\cdot)$ corresponds to the conductivity and reluctivity, respectively. Usually, they are piecewise constant due to different materials of which electrical devises are made. We mention that in non-conducting materials like air the conductivity is zero. We will also consider this practically important case that mathematically leads to a problem that is parabolic in the conducting regions and elliptic in non-conducting regions of the computational domain.

Time-periodic parabolic partial differential equations like (5) have been discussed, for example, in the works of Hackbusch [16], Vandewalle and Piessens [34, 35], Steuerwalt [32], Pao [29] and Lieberman [22, 23, 24]. As Steuerwalt comments in [32], the existence of a time-periodic solution implies the solvability of the corresponding initial value problem and vice versa. The unique solvability of the initial value problem, which corresponds to the state equation (5) is discussed and proved for example in Zeidler [37].

In order to determine the space-time variational formulation of the state equation (5), we need to define proper function spaces as used in Ladyzhenskaya et al. [21]. After that, we discuss existence and uniqueness of the solution of the state equation, and then of the optimal control problem.

Definition 1. *The Sobolev space $H^{1,0}(Q_T)$ is defined by*

$$H^{1,0}(Q_T) = \{y \in L^2(Q_T) : \nabla y \in [L^2(Q_T)]^3\},$$

where ∇ is the weak spatial gradient, and equipped with the norm

$$\|y\|_{H^{1,0}(Q_T)} = \left(\int_0^T \int_\Omega \left(y(\boldsymbol{x},t)^2 + \left|\nabla y(\boldsymbol{x},t)\right|^2\right) d\boldsymbol{x}\,dt\right)^{1/2}.$$

The following space includes the time derivative as well.

Definition 2. *The Sobolev space $H^{1,1}(Q_T)$ is defined by*

$$H^{1,1}(Q_T) = \{y \in L^2(Q_T) : \nabla y \in [L^2(Q_T)]^3, \frac{\partial y}{\partial t} \in L^2(Q_T)\},$$

where $\frac{\partial}{\partial t}$ is the weak time derivative, and equipped with the norm

$$\|y\|_{H^{1,1}(Q_T)} = \left(\int_0^T \int_\Omega \left(y(\boldsymbol{x},t)^2 + \left|\nabla y(\boldsymbol{x},t)\right|^2 + \left|\frac{\partial y}{\partial t}(\boldsymbol{x},t)\right|^2\right) d\boldsymbol{x}\,dt\right)^{1/2}.$$

Remark 1. *The space $H^{1,0}(Q_T)$ is equivalent to the space $L^2(0,T;H^1(\Omega))$ of abstract functions. The space $H^{1,1}(Q_T)$ is a subspace of $W_2^1(0,T;H^1(\Omega),L^2(\Omega)) := \{y \in L^2(0,T;H^1(\Omega)) : \frac{\partial}{\partial t}y \in L^2(0,T;(H^1(\Omega))^*)\}$.*

The following spaces include the boundary conditions and periodicity.

Definition 3. *For $k = 0, 1$, the Sobolev space $H_0^{1,k}(Q_T)$ is defined by*

$$H_0^{1,k}(Q_T) = \{y \in H^{1,k}(Q_T) : y = 0 \text{ on } \Sigma_T\},$$

whereas the Sobolev space $H_{0,per}^{1,1}(Q_T)$ is given by

$$H_{0,per}^{1,1}(Q_T) = \{y \in H_0^{1,1}(Q_T) : y(\boldsymbol{x}, 0) = y(\boldsymbol{x}, T) \ \text{ for almost all } \boldsymbol{x} \in \Omega\}.$$

The H-spaces introduced above are Hilbert spaces with scalar products corresponding to the norms in an obvious way. As we will see later, we can assume that the control u belongs to $H_{0,per}^{1,1}(Q_T)$. Hence, the control u is T-periodic like the state y. This a priori assumption will be clarified in Section 3. In order to derive the space-time variational formulation of the state equation (5), we multiply the parabolic partial differential equation with a test function $v \in H_{0,per}^{1,1}(Q_T)$, integrate over the space-time cylinder Q_T, and after integration by parts, we get the following variational formulation: Find $y \in H_{0,per}^{1,1}(Q_T)$ such that

$$\int_{Q_T} \left(\sigma(\boldsymbol{x}) \frac{\partial y}{\partial t} v + \nu(\boldsymbol{x}) \nabla y \cdot \nabla v \right) d\boldsymbol{x}\, dt = \int_{Q_T} u\, v \, d\boldsymbol{x}\, dt \qquad \forall v \in H_{0,per}^{1,1}(Q_T). \tag{6}$$

Since the functions y, u, v belong to $H_{0,per}^{1,1}(Q_T)$, we can expand these functions into Fourier series. Moreover, as a result of the multiharmonic representation of the desired state in the form

$$y_d(\boldsymbol{x}, t) = \sum_{k=0}^{N} \left(y_{dk}^c(\boldsymbol{x}) \cos(k\omega t) + y_{dk}^s(\boldsymbol{x}) \sin(k\omega t) \right), \tag{7}$$

where the Fourier coefficients are given by the formulas

$$y_{dk}^c(\boldsymbol{x}) = \frac{2}{T} \int_0^T y_d(\boldsymbol{x}, t) \cos(k\omega t)\, dt \ \text{ and } \ y_{dk}^s(\boldsymbol{x}) = \frac{2}{T} \int_0^T y(\boldsymbol{x}, t) \sin(k\omega t)\, dt, \tag{8}$$

the functions y, u, v will have a multiharmonic representation as well. For instance, for the state y, we have the representation

$$y(\boldsymbol{x}, t) = y_0^c(\boldsymbol{x}) + \sum_{k=1}^{N} y_k^c(\boldsymbol{x}) \cos(k\omega t) + y_k^s(\boldsymbol{x}) \sin(k\omega t) \tag{9}$$

with unknown Fourier coefficients $y_k^c(\boldsymbol{x})$ and $y_k^s(\boldsymbol{x})$. Here $\omega = \frac{2\pi}{T}$ and T denote the frequency and the periodicity, respectively. Inserting this multiharmonic ansatz into the space-time variational formulation (6) of the state equation and making use of the fact that the functions $\cos(k\omega t)$ and $\sin(k\omega t)$ are orthogonal with respect to the scalar product $(\cdot, \cdot)_{L^2(0,T)}$, we arrive at an equivalent variational formulation for determining the Fourier coefficients. Due to the linearity of the problem, this variational formulation decouples into separate variational problems for determining the Fourier coefficients $y_k^c(\boldsymbol{x})$ and $y_k^s(\boldsymbol{x})$ only corresponding to the k-th modes for $k = 0, 1, 2, \ldots, N$. Using the notation

$$\boldsymbol{v}_k = (v_k^c, v_k^s)^T, \quad \boldsymbol{v}_k^\perp = (-v_k^s, v_k^c)^T \quad \text{ and } \quad \nabla \boldsymbol{v}_k = (\nabla v_k^c, \nabla v_k^s)^T,$$

we arrive at the following variational formulation corresponding to every single mode $k = 1, 2, \ldots, N$: Find $\boldsymbol{y}_k \in \mathbb{V}$ such that

$$\underbrace{\int_\Omega \nu(\boldsymbol{x}) \nabla \boldsymbol{y}_k(\boldsymbol{x}) \cdot \nabla \boldsymbol{v}_k(\boldsymbol{x}) + k\omega\sigma(\boldsymbol{x}) \boldsymbol{y}_k(\boldsymbol{x}) \cdot \boldsymbol{v}_k^\perp(\boldsymbol{x})\, d\boldsymbol{x}}_{=:\langle A_k \boldsymbol{y}_k, \boldsymbol{v}_k \rangle} = \underbrace{\int_\Omega \boldsymbol{u}_k(\boldsymbol{x}) \cdot \boldsymbol{v}_k(\boldsymbol{x})\, d\boldsymbol{x}}_{=:\langle U_k, \boldsymbol{v}_k \rangle} \tag{10}$$

for all $\boldsymbol{v}_k \in \mathbb{V}$. The product $\langle \cdot, \cdot \rangle$ denotes the duality product $\langle \cdot, \cdot \rangle_{\mathbb{V}^* \times \mathbb{V}}$, where $\mathbb{V} := V \times V = (H_0^1(\Omega))^2$ denotes the space for the Fourier coefficients, and

$$V = H_0^1(\Omega) = \{y \in L^2(\Omega) : \nabla y \in L^2(\Omega) \text{ and } y = 0 \text{ on } \Gamma\}.$$

For ease of notation, the symbols $(\cdot,\cdot)_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$ as well as the symbols $(\cdot,\cdot)_{H^1(\Omega)}$ and $\|\cdot\|_{H^1(\Omega)}$ do not only indicate the scalar case but also the vector-valued case. We denote the L^2-inner product by

$$(\boldsymbol{y}_k, \boldsymbol{v}_k)_{L^2(\Omega)} = \sum_{j\in\{c,s\}} (y_k^j, v_k^j)_{L^2(\Omega)}$$

with the associated norm

$$\|\boldsymbol{y}_k\|^2_{L^2(\Omega)} = (\boldsymbol{y}_k, \boldsymbol{y}_k)_{L^2(\Omega)}$$

for our problem. The space $\mathbb{V} = (H_0^1(\Omega))^2$ is equipped with the norm

$$\|\boldsymbol{y}_k\|^2_{H^1(\Omega)} = \|\boldsymbol{y}_k\|^2_{L^2(\Omega)} + \|\nabla \boldsymbol{y}_k\|^2_{L^2(\Omega)}.$$

In addition, we obtain the variational formulation

$$\int_\Omega \nu(\boldsymbol{x}) \nabla y_0^c(\boldsymbol{x}) \cdot \nabla v_0^c(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega u_0^c(\boldsymbol{x}) v_0^c(\boldsymbol{x})\, d\boldsymbol{x} \tag{11}$$

for the mode $k = 0$, in the space $V = H_0^1(\Omega)$.

The variational problems (10) and (11) can be rewritten as operator equations: Find $\boldsymbol{y}_k \in \mathbb{V}$ such that

$$A_k \boldsymbol{y}_k = U_k \qquad \text{in } \mathbb{V}^*, \tag{12}$$

where the linear bounded operator $A_k \in L(\mathbb{V}, \mathbb{V}^*)$ and the linear bounded functional U_k are defined by the left-hand and right-hand sides of (10) or (11), respectively.

Theorem 1. *The variational problems (10) and (11) have a unique solution.*

Proof. Since $U_k \in \mathbb{V}^*$ and the bilinear form $\langle A_k \cdot, \cdot\rangle$ is bounded and coercive on $\mathbb{V}$, unique solvability of the variational formulation (10) follows from the theorem of Lax-Milgram. The same argument can be used to prove unique solvability for the variational problem (11) in the space V. □

Now we are in the position to prove existence and uniqueness of a solution of the variational problem (6) corresponding to our state equation (5).

Theorem 2. *The space-time variational problem (6) has a unique solution.*

Proof. The right hand side u and the solution y have a multiharmonic representation. Since existence and uniqueness of a solution for the problems corresponding to the modes $k = 0, 1, 2, \ldots, N$ have already been showed in Theorem 1, we obtain existence and uniqueness of the whole variational problem (6) as well, and the multiharmonic representation of the solution y given by (9) can be uniquely determined by inserting all solutions of the variational problems (10) for $k = 1, 2, \ldots, N$ and (11) for $k = 0$, which are the Fourier coefficients of y, into (9). □

We mention that (5) must always be understood in the weak sense, more precisely, in the sense of (6).

3. The Optimality System

In order to determine the unique weak solution of the optimal control problem (4)-(5), we derive its optimality system from the corresponding Lagrange functional. The Lagrange functional of (4)-(5) is given by

$$\mathcal{L}(y, u, p) := \mathcal{J}(y, u) - \int_0^T \int_\Omega \Big(\sigma \frac{\partial y}{\partial t} - \nabla \cdot \big(\nu \nabla y\big) - u\Big) p\, d\boldsymbol{x}\, dt, \tag{13}$$

where p is the co-state (Lagrange multiplier). A stationary point (y, u, p) of the Lagrange functional is characterized by the following three conditions:

$$\begin{cases} \nabla_y \mathcal{L}(y,u,p) = 0, \\ \nabla_u \mathcal{L}(y,u,p) = 0, \\ \nabla_p \mathcal{L}(y,u,p) = 0, \end{cases} \tag{14}$$

called optimality system, see, e.g., [33]. We can eliminate the control u from the optimality system (14) using the second condition, i.e.

$$u = -\lambda^{-1} p \quad \text{in } Q_T, \tag{15}$$

i.e. u corresponds in all but multiplication with $-\lambda^{-1}$ to the co-state p. Hence, it appears very natural to choose y, p and also u from the same space as it was done in Section 2. By (13), (14) and (15), we obtain the following (classical) formulation of the optimality system:

$$\begin{cases} \sigma(\boldsymbol{x})\dfrac{\partial}{\partial t} y(\boldsymbol{x},t) - \nabla\cdot(\nu(\boldsymbol{x})\nabla y(\boldsymbol{x},t)) = -\lambda^{-1}p(\boldsymbol{x},t) & \forall(\boldsymbol{x},t)\in Q_T, \\ y(\boldsymbol{x},t) = 0 & \forall(\boldsymbol{x},t)\in\Sigma_T, \\ y(\boldsymbol{x},0) = y(\boldsymbol{x},T) & \forall\boldsymbol{x}\in\Omega, \\ -\sigma(\boldsymbol{x})\dfrac{\partial}{\partial t} p(\boldsymbol{x},t) - \nabla\cdot(\nu(\boldsymbol{x})\nabla p(\boldsymbol{x},t)) = y(\boldsymbol{x},t) - y_d(\boldsymbol{x},t) & \forall(\boldsymbol{x},t)\in Q_T, \\ p(\boldsymbol{x},t) = 0 & \forall(\boldsymbol{x},t)\in\Sigma_T, \\ p(\boldsymbol{x},T) = p(\boldsymbol{x},0) & \forall\boldsymbol{x}\in\Omega. \end{cases} \tag{16}$$

The corresponding weak formulation is given as follows: Let $y_d \in H^{0,1}_{0,per}(Q_T)$ be given. Find y and p from $H^{1,1}_{0,per}(Q_T)$ such that

$$\begin{cases} \displaystyle\int_0^T\int_\Omega y\,v - \nu(\boldsymbol{x})\nabla p\cdot\nabla v + \sigma(\boldsymbol{x})\frac{\partial}{\partial t}p\,v\,d\boldsymbol{x}\,dt = \int_0^T\int_\Omega y_d\,v\,d\boldsymbol{x}\,dt, \\ \displaystyle\int_0^T\int_\Omega \nu(\boldsymbol{x})\nabla y\cdot\nabla q + \sigma(\boldsymbol{x})\frac{\partial}{\partial t}y\,q + \lambda^{-1}p\,q\,d\boldsymbol{x}\,dt = 0 \end{cases} \tag{17}$$

for all test functions $v, q \in H^{1,1}_{0,per}(Q_T)$.

4. Multiharmonic Finite Element Discretization

Let us recall that we have assumed that desired state $y_d(\boldsymbol{x},t)$ has a multiharmonic representation of the form (7) with the Fourier coefficients (8). We mention here that, in general, we only have an approximation of the desired state $y_d(\boldsymbol{x},t)$ by a truncated Fourier series of the form (7). However here we suppose that the given desired state is really multiharmonic that is often the case in electromagnetics.

Due to the linearity of the optimality system (17) the state y and the co-state p are multharmonic as well, i.e. they can be represented in the same form as the given desired state y_d:

$$\begin{aligned} y(\boldsymbol{x},t) &= \sum_{k=0}^{N}\left(y_k^c(\boldsymbol{x})\cos(k\omega t) + y_k^s(\boldsymbol{x})\sin(k\omega t)\right), \\ p(\boldsymbol{x},t) &= \sum_{k=0}^{N}\left(p_k^c(\boldsymbol{x})\cos(k\omega t) + p_k^s(\boldsymbol{x})\sin(k\omega t)\right) \end{aligned} \tag{18}$$

with unknown Fourier coefficients $\boldsymbol{y}_k = (y_k^c, y_k^s)$ and $\boldsymbol{p}_k = (p_k^c, p_k^s)$. In order to define these unknown Fourier coefficients, we insert (7) and (18) into the optimality system

(17) and test with all modes up to the number N. Using the orthogonality of the sine and cosine functions and, of course, the linearity of our equations, we arrive at the following optimality equations for the Fourier coefficients corresponding to the modes $k = 1, 2, ..., N$: Find $\boldsymbol{y}_k, \boldsymbol{p}_k \in \mathbb{V} = V \times V$ such that

$$(19) \quad \begin{cases} \int_\Omega \boldsymbol{y}_k \cdot \boldsymbol{v}_k - \nu(\boldsymbol{x})\nabla \boldsymbol{p}_k \cdot \nabla \boldsymbol{v}_k + k\omega\sigma(\boldsymbol{x})\boldsymbol{p}_k \cdot \boldsymbol{v}_k^\perp \, d\boldsymbol{x} = \int_\Omega \boldsymbol{y_d}_k \cdot \boldsymbol{v}_k \, d\boldsymbol{x}, \\ \int_\Omega \nu(\boldsymbol{x})\nabla \boldsymbol{y}_k \cdot \nabla \boldsymbol{q}_k + k\omega\sigma(\boldsymbol{x})\boldsymbol{y}_k \cdot \boldsymbol{q}_k^\perp + \lambda^{-1}\boldsymbol{p}_k \cdot \boldsymbol{q}_k \, d\boldsymbol{x} = 0 \end{cases}$$

for all test functions $\boldsymbol{v}_k, \boldsymbol{q}_k \in \mathbb{V} = V \times V$. In the case of $k = 0$, we obtain the optimality system: Find $y_0^c \in V = H_0^1(\Omega)$ and $p_0^c \in V = H_0^1(\Omega)$ such that

$$(20) \quad \begin{cases} \int_\Omega y_0^c \cdot v_0^c - \nu(\boldsymbol{x})\nabla p_0^c \cdot \nabla v_0^c \, d\boldsymbol{x} = \int_\Omega y_{d0}^c \cdot v_0^c \, d\boldsymbol{x}, \quad \forall v_0^c \in V = H_0^1(\Omega) \\ \int_\Omega \nu(\boldsymbol{x})\nabla y_0^c \cdot \nabla q_0^c + \lambda^{-1} p_0^c \cdot q_0^c \, d\boldsymbol{x} = 0, \quad \forall q_0^c \in V = H_0^1(\Omega). \end{cases}$$

Now we approximate the unknown Fourier coefficients $\boldsymbol{y}_k = (y_k^c, y_k^s), \boldsymbol{p}_k = (p_k^c, p_k^s) \in \mathbb{V}$ by finite element functions $\boldsymbol{y}_{kh} = (y_{kh}^c, y_{kh}^s), \boldsymbol{p}_{kh} = (p_{kh}^c, p_{kh}^s)$ from some finite element subspace $\mathbb{V}_h = V_h \times V_h$ of $\mathbb{V}$. Here and in the following, h denotes the usual discretization parameter such that $n = n_h = \dim V_h = O(h^{-d})$. In order to simplify the notation we sometimes omit h as an index as it is indicated above and below. Let $\{\varphi_i(x) = \varphi_{ih}(x) : i = 1, 2, ..., n_h\}$ be the standard nodal basis of $V_h = \text{span}\{\varphi_1, ..., \varphi_n\}$. In our numerical experiments, we have used continuous, piecewise linear finite elements on triangles (2d) and tetrahedrons (3d) on a regular triangulation to construct the finite element subspace V_h and its basis, see Ciarlet [9]. Once the basis is chosen, the variational formulation (19) on V_h yields the linear system

$$(21) \quad \begin{pmatrix} M_h & 0 & -K_h & k\omega M_{h,\sigma} \\ 0 & M_h & -k\omega M_{h,\sigma} & -K_h \\ -K_h & -k\omega M_{h,\sigma} & -\lambda^{-1} M_h & 0 \\ k\omega M_{h,\sigma} & -K_h & 0 & -\lambda^{-1} M_h \end{pmatrix} \begin{pmatrix} \underline{y}_k^c \\ \underline{y}_k^s \\ \underline{p}_k^c \\ \underline{p}_k^s \end{pmatrix} = \begin{pmatrix} M_h \underline{y}_{dk}^c \\ M_h \underline{y}_{dk}^s \\ 0 \\ 0 \end{pmatrix},$$

for defining the nodal parameter vectors $\underline{y}_k^j = \underline{y}_{kh}^j = (y_{k,i}^j)_{i=1,...,n} \in \mathbb{R}^n$ and $\underline{p}_k^j = \underline{p}_{kh}^j = (p_{k,i}^j)_{i=1,...,n} \in \mathbb{R}^n$ of the finite element approximations

$$(22) \quad y_{kh}^j(x) = \sum_{i=1}^n y_{k,i}^j \varphi_i(x) \quad \text{and} \quad p_{kh}^j(x) = \sum_{i=1}^n p_{k,i}^j \varphi_i(x)$$

to the Fourier coefficients $y_k^j(x)$ and $p_k^j(x)$ with $j \in \{c, s\}$. The matrix M_h corresponds to the mass matrix, $M_{h,\sigma}$ to the weighted mass matrix, and K_h corresponds to the stiffness matrix, the entries of which are computed by the formulas

$$M_h^{ij} = \int_\Omega \varphi_i \varphi_j \, d\boldsymbol{x} \qquad M_{h,\sigma}^{ij} = \int_\Omega \sigma \varphi_i \varphi_j \, d\boldsymbol{x} \qquad K_h^{ij} = \int_\Omega \nu \nabla \varphi_i \cdot \nabla \varphi_j \, d\boldsymbol{x},$$

where $i, j = 1, ..., n$. For $k = 0$, we obtain the following system of linear equations:

$$(23) \quad \begin{pmatrix} M_h & -K_h \\ -K_h & -\lambda^{-1} M_h \end{pmatrix} \begin{pmatrix} \underline{y}_0^c \\ \underline{p}_0^c \end{pmatrix} = \begin{pmatrix} M_h \underline{y}_{d0}^c \\ 0 \end{pmatrix}.$$

Once the linear systems (21) and (23) are solved, we can easily construct the finite element approximations

$$\begin{aligned} y_h(\boldsymbol{x},t) &= \sum_{k=0}^{N} \left(y_{kh}^c(\boldsymbol{x}) \cos(k\omega t) + y_{kh}^s(\boldsymbol{x}) \sin(k\omega t) \right), \\ p_h(\boldsymbol{x},t) &= \sum_{k=0}^{N} \left(p_{kh}^c(\boldsymbol{x}) \cos(k\omega t) + p_{kh}^s(\boldsymbol{x}) \sin(k\omega t) \right) \end{aligned} \tag{24}$$

to the state $y(\boldsymbol{x},t)$ and to the co-state $p(\boldsymbol{x},t)$ given by the multiharmonic representation (18). The finite element approximation $u_h(\boldsymbol{x},t)$ to the control u is given by the formula $u_h = -\lambda^{-1} p_h$.

Before we analyze the discretization errors $y - y_h$ and $p - p_h$ in Section 6, we construct a fast and robust preconditioned MINRES method for the iterative solutions of systems (21) and (23) in the next section. In fact, this is a challenging and at the same time practically important problem since the condition number of the system matrices is affected by the discretization and problem parameters h, N, ω, ν, σ and λ in a very bad way. The convergence rate of the MINRES method as well as of other iterative methods depends on the spectrum of the iteration matrix. Therefore, the construction of a preconditioner that eliminates this dependence of the spectrum from the "bad" parameters is of great importance in practical computations. We consider the construction of preconditioners for the system matrices of our systems (21) and (23), which yield robust (independent of all bad parameters) convergence rates, as the main result of our paper.

5. A Robust Preconditioner for the MINRES Solver

The resulting linear system (21) is a saddle point problem of the form

$$\underbrace{\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}}_{=:\mathcal{A}} \begin{pmatrix} \underline{y} \\ \underline{p} \end{pmatrix} = \underbrace{\begin{pmatrix} \underline{f} \\ 0 \end{pmatrix}}_{=:\underline{f}}, \tag{25}$$

where the matrices A and C are symmetric and, even, positive definite. Therefore, it can be solved by a preconditioned MINRES method, see Paige and Saunders [28]. The goal of this section is to construct preconditioners, which yield robust and fast convergence for the preconditioned MINRES method. The construction of efficient preconditioners is subject of discussion in many papers. For example, Hiptmair [17] employs operator preconditioning for constructing preconditioners for discrete linear operators coming from a Galerkin approach and presents applications as for example saddle point problems to finite and boundary elements. Mardal and Winther [26] also discuss an abstract approach constructing preconditioners for symmetric linear systems in a Hilbert space setting and the application of this theory on systems of partial differential equations which correspond to saddle point problems.

In Subsection 5.1, we start with an easier case by assuming that the parameter σ is constant. We construct preconditioners following the strategy presented in Zulehner [38], which is based on space interpolation theory. Motivated by the resulting preconditioner, we choose an initial guess for a preconditioner in the more general case of σ being piecewise constant, which is discussed in subsection 5.2. By introducing proper parameter dependent norms, we verify the assumptions of the theorem of Babuška-Aziz established in Babuška [3] and Babuška and Aziz [4], which finally yields a parameter robust convergence rate as desired. This procedure leads to a block diagonal preconditioner, which might be not efficient in practice. In subsection 5.3, we briefly discuss the construction of practical preconditioners.

As Mardal and Winther [26] suggest, the blocks of the preconditioner have to be replaced by operators which are more cost efficient but have equivalent mapping properties. They are often constructed by multigrid methods, domain decomposition methods or similar techniques.

5.1. **Preconditioning by Operator Interpolation.** In the first subsection, we start with a special case by assuming that σ is constant. Hence, in this case, we have $M_{h,\sigma} = \sigma M_h$. The linear system (25) is then given by the block matrices

$$A := \begin{pmatrix} M_h & 0 \\ 0 & M_h \end{pmatrix}, \; B := \begin{pmatrix} -K_h & -k\omega\sigma M_h \\ k\omega\sigma M_h & -K_h \end{pmatrix}, \; C := \lambda^{-1} A$$

and the vectors

$$\underline{f} := \begin{pmatrix} M_h \underline{y}^c_{dk} \\ M_h \underline{y}^s_{dk} \end{pmatrix}, \; \underline{y} := \begin{pmatrix} \underline{y}^c_k \\ \underline{y}^s_k \end{pmatrix}, \; \underline{p} := \begin{pmatrix} \underline{p}^c_k \\ \underline{p}^s_k \end{pmatrix}.$$

As mentioned above, the symmetric and indefinite linear system (25) can be solved by a preconditioned MINRES method. A convergence result for the preconditioned MINRES Method can be found in Greenbaum [13]. It states that the convergence rate of the preconditioned MINRES method depends on the condition number of the preconditioned system. This convergence result is summarized in the following theorem in detail.

Theorem 3. *The preconditioned MINRES method applied to the system $\mathcal{A}x = f$ with some symmetric and positive definite (SPD) preconditioner $\mathcal{P}$ converges to the solution of this system for an arbitrary initial guess x_0. More precisely, the preconditioned residual $r_m = \mathcal{P}^{-1}(f - \mathcal{A}x_m)$ after m iterations can be estimated by the initial residual r_0 as follows:*

$$\|r_{2m}\|_{\mathcal{P}} \le \frac{2q^m}{1+q^{2m}} \|r_0\|_{\mathcal{P}}, \tag{26}$$

where

$$q = \frac{\kappa_{\mathcal{P}}(\mathcal{P}^{-1}\mathcal{A}) - 1}{\kappa_{\mathcal{P}}(\mathcal{P}^{-1}\mathcal{A}) + 1}$$

with the condition number $\kappa_{\mathcal{P}}(\mathcal{P}^{-1}\mathcal{A}) := \|\mathcal{P}^{-1}\mathcal{A}\|_{\mathcal{P}} \|(\mathcal{P}^{-1}\mathcal{A})^{-1}\|_{\mathcal{P}}$ of the preconditioned system matrix, the $\mathcal{P}$-energy norm $\|\cdot\|_{\mathcal{P}} = (\mathcal{P}\cdot,\cdot)^{1/2}$, and the corresponding matrix norm $\|\cdot\|_{\mathcal{P}}$.

Proof. See Greenbaum [13]. □

Hence, we are going to construct preconditioners for the preconditioned MINRES method such that the condition number $\kappa_{\mathcal{P}}(\mathcal{P}^{-1}\mathcal{A})$ of the preconditioned system $\mathcal{P}^{-1}\mathcal{A}$ is independent of all "bad" parameters, i.e. h, N, ω, λ, ν and σ. In order to obtain parameter robust convergence rates, we first construct block diagonal preconditioners by the operator (matrix) interpolation technique presented in Zulehner [38].

Theorem 4. *Let A and C be symmetric and positive definite $N \times N$ matrices and let*

$$S = C + BA^{-1}B^T \qquad \text{and} \qquad R = A + B^T C^{-1} B$$

be the negative Schur complements. If $\mathcal{A}$ is preconditioned by

$$\mathcal{P}_0 = \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix} \qquad \text{or} \qquad \mathcal{P}_1 = \begin{pmatrix} R & 0 \\ 0 & C \end{pmatrix}, \tag{27}$$

then the eigenvalues of the preconditioned matrices $\mathcal{P}_0^{-1}\mathcal{A}$ and $\mathcal{P}_1^{-1}\mathcal{A}$ are located in the set $(-1, \frac{1-\sqrt{5}}{2}] \cup \{1\} \cup (1, \frac{1+\sqrt{5}}{2}]$.

Proof. See Kuznetsov [20] and Murphy et al. [27]. □

Theorem 4 immediately yields the following norm estimates.

Corollary 1. *The inequalities*

$$\underline{c}\,\|x\|_{\mathcal{P}_0} \le \|\mathcal{A}x\|_{\mathcal{P}_0^{-1}} \le \overline{c}\,\|x\|_{\mathcal{P}_0} \quad \text{and} \quad \underline{c}\,\|x\|_{\mathcal{P}_1} \le \|\mathcal{A}x\|_{\mathcal{P}_1^{-1}} \le \overline{c}\,\|x\|_{\mathcal{P}_1} \tag{28}$$

are valid for all $x \in \mathbb{R}^{2N}$*, with* $\underline{c} = (\sqrt{5}-1)/2$ *and* $\overline{c} = (\sqrt{5}+1)/2$.

In our model problem, the negative Schur complements are given by

$$S = \begin{pmatrix} K_h M_h^{-1} K_h + (k^2\omega^2\sigma^2 + \lambda^{-1})M_h & 0 \\ 0 & K_h M_h^{-1} K_h + (k^2\omega^2\sigma^2 + \lambda^{-1})M_h \end{pmatrix}$$

and

$$R = \begin{pmatrix} \lambda K_h M_h^{-1} K_h + (k^2\omega^2\sigma^2\lambda + 1)M_h & 0 \\ 0 & \lambda K_h M_h^{-1} K_h + (k^2\omega^2\sigma^2\lambda + 1)M_h \end{pmatrix}.$$

Hence, $R = \lambda S$. In practice, it is hard to work with these Schur complements. The idea is to apply the following operator interpolation theorem, which is based on the construction of intermediate spaces via the so called real method, which includes the J- and the K-method. The idea of these methods is due to Lions and Peetre and the theory of the real method is developed e.g. in Bergh and Löfström [8], see Adams and Fournier [2].

Theorem 5. *Let* $\mathcal{A} : \mathbb{R}^{2N} \longrightarrow \mathbb{R}^{2N}$ *with*

$$\|\mathcal{A}x\|_{Y_0} \le c_0 \|x\|_{X_0} \qquad \text{and} \qquad \|\mathcal{A}x\|_{Y_1} \le c_1 \|x\|_{X_1},$$

where the norms $\|\cdot\|_{X_i}$ *and* $\|\cdot\|_{Y_i}$ *are the norms associated to the inner products*

$$(x,y)_{X_i} = \langle M_i x, y\rangle \qquad \text{and} \qquad (x,y)_{Y_i} = \langle N_i x, y\rangle$$

with symmetric positive definite matrices $M_0, M_1, N_0, N_1 \in \mathbb{R}^{2N\times 2N}$*. Then, for* $X_\theta = [X_0, X_1]_\theta$ *and* $Y_\theta = [Y_0, Y_1]_\theta$ *with* $\theta \in [0,1]$*, we have*

$$\|\mathcal{A}x\|_{Y_\theta} \le c_0^{1-\theta} c_1^{\theta} \|x\|_{X_\theta}. \tag{29}$$

The norms $\|\cdot\|_{X_\theta}$ *and* $\|\cdot\|_{Y_\theta}$ *are the norms associated to the inner products*

$$(x,y)_{X_\theta} = \langle M_\theta x, y\rangle \quad \text{with} \quad M_\theta = [M_0, M_1]_\theta = M_0^{1/2}\left(M_0^{-1/2} M_1 M_0^{-1/2}\right)^\theta M_0^{1/2},$$

$$(x,y)_{Y_\theta} = \langle N_\theta x, y\rangle \quad \text{with} \quad N_\theta = [N_0, N_1]_\theta = N_0^{1/2}\left(N_0^{-1/2} N_1 N_0^{-1/2}\right)^\theta N_0^{1/2}.$$

Proof. See Adams and Fournier [2]. □

Hence, from interpolating between the block diagonal preconditioners $\mathcal{P}_0$ and $\mathcal{P}_1$, we can obtain again parameter independent condition number estimates for all $\theta \in [0,1]$. In our case, $M_0 = \mathcal{P}_0$, $M_1 = \mathcal{P}_1$, $N_0 = \mathcal{P}_0^{-1}$ and $N_1 = \mathcal{P}_1^{-1}$. We choose $\theta = \frac{1}{2}$ and obtain the block diagonal matrix

$$\mathcal{P}_{1/2} = \begin{pmatrix} [A,R]_{1/2} & 0 \\ 0 & [S,C]_{1/2} \end{pmatrix} \tag{30}$$

with

$$[A,R]_{1/2} = A^{1/2}(A^{-1/2}RA^{-1/2})^{1/2}A^{1/2}$$

and

$$[S,C]_{1/2} = S^{1/2}(S^{-1/2}CS^{-1/2})^{1/2}S^{1/2}.$$

In the following, we use the special notation $A \sim B$ for the spectral equivalence of the matrices A and B. Two SPD matrices A and B in $\mathbb{R}^{n\times n}$ are called spectral equivalent, denoted by $A \sim B$, if there exist positive constants $\underline{c}$ and $\overline{c}$ which are independent of all involved parameters such that

$$\underline{c}\, x^T A\, x \leq x^T B\, x \leq \overline{c}\, x^T A\, x$$

for all $x \in \mathbb{R}^n$.

Since A and R are block diagonal, the diagonal entries $[A, R]_{1/2}^{(1,1)}$ and $[A, R]_{1/2}^{(2,2)}$ can be estimated as follows:

$$\begin{aligned}
[A, R]_{1/2}^{(1,1)} =&[A, R]_{1/2}^{(2,2)} \\
=&M_h^{1/2}(\lambda M_h^{-1/2} K_h M_h^{-1} K_h M_h^{-1/2} + (k^2\omega^2\sigma^2\lambda + 1)I)^{1/2} M_h^{1/2} \\
\sim&\sqrt{\lambda} M_h^{1/2} (M_h^{-1/2} K_h M_h^{-1} K_h M_h^{-1/2})^{1/2} M_h^{1/2} \\
&+ \sqrt{k^2\omega^2\sigma^2\lambda + 1}\, M_h^{1/2} M_h^{1/2} \\
=&\sqrt{\lambda} M_h^{1/2} (M_h^{-1/2} K_h M_h^{-1/2}) M_h^{1/2} + \sqrt{k^2\omega^2\sigma^2\lambda + 1}\, M_h \\
=&\sqrt{\lambda} K_h + \sqrt{k^2\omega^2\sigma^2\lambda + 1}\, M_h \\
\sim&\sqrt{\lambda} K_h + (k\omega\sigma\sqrt{\lambda} + 1)\, M_h =: D,
\end{aligned}$$

where we used the inequality $1 + x^2 \leq (1 + x)^2 \leq 2(1 + x^2)$ that is valid for all reals $x \geq 0$ as well as $I + X^2 \leq (I + X)^2 \leq 2(I + X^2)$ is valid for non-negative symmetric matrices X. Analogously, since $S = \lambda^{-1}R$ and $C = \lambda^{-1}A$, we have

$$\begin{aligned}
[S, C]_{1/2}^{(1,1)} &= [S, C]_{1/2}^{(2,2)} \\
&= [\lambda^{-1}R, \lambda^{-1}A]_{1/2}^{(1,1)} \\
&= \lambda^{-1}[A, R]_{1/2}^{(1,1)} \\
&\sim \lambda^{-1}(\sqrt{\lambda} K_h + (k\omega\sigma\sqrt{\lambda} + 1) M_h) = \lambda^{-1} D.
\end{aligned}$$

Thus, we have obtained a new block diagonal preconditioner for the MINRES solver of the problem (21) which is given by

$$\mathcal{P}_{1/2} = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & \lambda^{-1}D & 0 \\ 0 & 0 & 0 & \lambda^{-1}D \end{pmatrix}. \tag{31}$$

This block diagonal preconditioner is much easier to realize in practice than the previous preconditioners $\mathcal{P}_0$ and $\mathcal{P}_1$. Due to Theorems 4 and 5, we obtain the estimates

$$\underline{c}\, \|x\|_{\mathcal{P}_{1/2}} \leq \|\mathcal{A}x\|_{\mathcal{P}_{1/2}^{-1}} \leq \overline{c}\, \|x\|_{\mathcal{P}_{1/2}}, \tag{32}$$

which yield a robust estimate of the condition number

$$\kappa_{\mathcal{P}_{1/2}}(\mathcal{P}_{1/2}^{-1}\mathcal{A}) \leq \overline{c}/\underline{c}$$

with constants $\underline{c} = (\sqrt{5} - 1)/2$ and $\overline{c} = (\sqrt{5} + 1)/2$ independent of all involved parameters including the meshsize h. Therefore, Theorem 3 leads to robust convergence rates of the MINRES method.

5.2. **Preconditioning in Case of Piecewise Constant σ.** We now assume that σ is only piecewise constant. Moreover, we allow that σ is zero in some regions of the computational domain Ω. This situation is typical in electromagnetics where σ

is nothing but the conductivity that is zero in non-conducting regions. The system matrix $\mathcal{A}_k$ of (21) is now given by

$$\mathcal{A} := \begin{pmatrix} M_h & 0 & -K_h & k\omega M_{h,\sigma} \\ 0 & M_h & -k\omega M_{h,\sigma} & -K_h \\ -K_h & -k\omega M_{h,\sigma} & -\lambda^{-1} M_h & 0 \\ k\omega M_{h,\sigma} & -K_h & 0 & -\lambda^{-1} M_h \end{pmatrix}, \tag{33}$$

where we again omit the mode number k. Since now $M_{h,\sigma} \neq \sigma M_h$, we cannot apply the operator interpolation theory in this case ! However, we get an inspiration for choosing a suitable block diagonal preconditioner according to the block diagonal preconditioner $\mathcal{P}_{1/2}$ from the previous section. Replacing σM_h by $M_{h,\sigma}$ in (31), we arrive at the new preconditioner

$$\mathcal{P} = \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & \lambda^{-1} D & 0 \\ 0 & 0 & 0 & \lambda^{-1} D \end{pmatrix}, \tag{34}$$

where $D = \sqrt{\lambda} K_h + k\omega\sqrt{\lambda} M_{h,\sigma} + M_h$. This preconditioner $\mathcal{P}$ is now our candidate for a robust preconditioner for the system matrix $\mathcal{A}$. In order to obtain robust norm estimates for the preconditioned system matrix $\mathcal{P}^{-1}\mathcal{A}$, we look at Babuška-Aziz' theorem and see that the norm estimates which we have to prove are equivalent to the assumptions (inf-sup- and sup-sup-conditions) in the theorem of Babuška-Aziz that, at the same time, provides existence, uniqueness, and a priori estimates. Moreover, the assumptions of Babuška-Aziz' theorem also yield discretization error estimates which we are going to present in Section 6.

Let us return to the variational formulation (19) of the optimality system for each mode k, and let us define the corresponding bilinear form

$$\begin{aligned} \mathcal{B}((\boldsymbol{y}_k, \boldsymbol{p}_k), (\boldsymbol{v}_k, \boldsymbol{q}_k)) := & \int_\Omega \boldsymbol{y}_k \cdot \boldsymbol{v}_k - \nu \nabla \boldsymbol{p}_k \cdot \nabla \boldsymbol{v}_k + k\omega\sigma \boldsymbol{p}_k \cdot \boldsymbol{v}_k^\perp \, d\boldsymbol{x} \\ & + \int_\Omega \nu \nabla \boldsymbol{y}_k \cdot \nabla \boldsymbol{q}_k + k\omega\sigma \boldsymbol{y}_k \cdot \boldsymbol{q}_k^\perp + \lambda^{-1} \boldsymbol{p}_k \cdot \boldsymbol{q}_k \, d\boldsymbol{x}. \end{aligned} \tag{35}$$

Hence, the variational problem (19) reads now as follows: Find $(\boldsymbol{y}_k, \boldsymbol{p}_k) \in \mathbb{V}^2 = (H_0^1(\Omega))^4$ such that

$$\mathcal{B}((\boldsymbol{y}_k, \boldsymbol{p}_k), (\boldsymbol{v}_k, \boldsymbol{q}_k)) = \int_\Omega \boldsymbol{y}_{\boldsymbol{d}k} \cdot \boldsymbol{v}_k \, d\boldsymbol{x} \tag{36}$$

for all test functions $(\boldsymbol{v}_k, \boldsymbol{q}_k) \in \mathbb{V}^2$. The initial guess (34) for the preconditioner $\mathcal{P}$ yields the following definitions of inner products and associated norms. We first define a non-standard (weighted) inner product in $\mathbb{V} = (H_0^1(\Omega))^2$ by

$$(\boldsymbol{y}_k, \boldsymbol{v}_k)_{\mathbb{V}} = \sqrt{\lambda}\, (\nu \nabla \boldsymbol{y}_k, \nabla \boldsymbol{v}_k)_{L^2(\Omega)} + k\omega\sqrt{\lambda}\, (\sigma \boldsymbol{y}_k, \boldsymbol{v}_k)_{L^2(\Omega)} + (\boldsymbol{y}_k, \boldsymbol{v}_k)_{L^2(\Omega)}.$$

The associated norm is then given by

$$\|\boldsymbol{y}_k\|_{\mathbb{V}}^2 = \sqrt{\lambda}\, (\nu \nabla \boldsymbol{y}_k, \nabla \boldsymbol{y}_k)_{L^2(\Omega)} + k\omega\sqrt{\lambda}\, (\sigma \boldsymbol{y}_k, \boldsymbol{y}_k)_{L^2(\Omega)} + \|\boldsymbol{y}_k\|_{L^2(\Omega)}^2,$$

which differs from the standard H^1-norms. Finally, we define an inner product in $\mathbb{V}^2 = (H_0^1(\Omega))^4$ by

$$((\boldsymbol{y}_k, \boldsymbol{p}_k), (\boldsymbol{v}_k, \boldsymbol{q}_k))_{\mathcal{P}} = (\boldsymbol{y}_k, \boldsymbol{v}_k)_{\mathbb{V}} + \lambda^{-1} (\boldsymbol{p}_k, \boldsymbol{q}_k)_{\mathbb{V}}.$$

The associated norm is given by

$$\|(\boldsymbol{y}_k, \boldsymbol{p}_k)\|_{\mathcal{P}}^2 = \|\boldsymbol{y}_k\|_{\mathbb{V}}^2 + \lambda^{-1} \|\boldsymbol{p}_k\|_{\mathbb{V}}^2.$$

Next, we verify the assumptions (inf-sup- and sup-sup-conditions) of the theorem of Babuška-Aziz.

Theorem 6. *The following inequalities are valid:*

$$\underline{c}\,\|(\boldsymbol{y}_k,\boldsymbol{p}_k)\|_{\mathcal{P}} \le \sup_{0\neq(\boldsymbol{v}_k,\boldsymbol{q}_k)\in\mathbb{V}^2} \frac{\mathcal{B}((\boldsymbol{y}_k,\boldsymbol{p}_k),(\boldsymbol{v}_k,\boldsymbol{q}_k))}{\|(\boldsymbol{v}_k,\boldsymbol{q}_k)\|_{\mathcal{P}}} \le \overline{c}\,\|(\boldsymbol{y}_k,\boldsymbol{p}_k)\|_{\mathcal{P}} \tag{37}$$

for all $(\boldsymbol{y}_k,\boldsymbol{p}_k)\in\mathbb{V}^2$ *with constants* $\underline{c}=1/\sqrt{3}$ *and* $\overline{c}=1$.

Proof. We start with the proof of the inequality from above. Due to the triangle inequality, it follows that

$$\begin{aligned}\big|\mathcal{B}((\boldsymbol{y}_k,\boldsymbol{p}_k),(\boldsymbol{v}_k,\boldsymbol{q}_k))\big| \le{}& \Big|\int_\Omega \boldsymbol{y}_k\cdot\boldsymbol{v}_k\,d\boldsymbol{x}\Big| + \Big|\int_\Omega \nu\nabla\boldsymbol{p}_k\cdot\nabla\boldsymbol{v}_k\,d\boldsymbol{x}\Big| \\ &+ \Big|\int_\Omega k\omega\,\sigma\boldsymbol{p}_k\cdot\boldsymbol{v}_k^\perp\,d\boldsymbol{x}\Big| + \Big|\int_\Omega \nu\nabla\boldsymbol{y}_k\cdot\nabla\boldsymbol{q}_k\,d\boldsymbol{x}\Big| \\ &+ \Big|\int_\Omega k\omega\,\sigma\boldsymbol{y}_k\cdot\boldsymbol{q}_k^\perp\,d\boldsymbol{x}\Big| + \Big|\int_\Omega \lambda^{-1}\boldsymbol{p}_k\cdot\boldsymbol{q}_k\,d\boldsymbol{x}\Big|.\end{aligned}$$

After appropriate scaling with the parameter λ and applying several times the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\big|\mathcal{B}((\boldsymbol{y}_k,\boldsymbol{p}_k),(\boldsymbol{v}_k,\boldsymbol{q}_k))\big| \le{}& \Big|\int_\Omega \boldsymbol{y}_k\cdot\boldsymbol{v}_k\,d\boldsymbol{x}\Big| + \Big|\int_\Omega \nu\lambda^{-1/4}\nabla\boldsymbol{p}_k\cdot\lambda^{1/4}\nabla\boldsymbol{v}_k\,d\boldsymbol{x}\Big| \\ &+ \Big|\int_\Omega k\omega\,\sigma\lambda^{-1/4}\boldsymbol{p}_k\cdot\lambda^{1/4}\boldsymbol{v}_k^\perp\,d\boldsymbol{x}\Big| + \Big|\int_\Omega \nu\lambda^{-1/4}\nabla\boldsymbol{y}_k\cdot\lambda^{1/4}\nabla\boldsymbol{q}_k\,d\boldsymbol{x}\Big| \\ &+ \Big|\int_\Omega k\omega\,\sigma\lambda^{-1/4}\boldsymbol{y}_k\cdot\lambda^{1/4}\boldsymbol{q}_k^\perp\,d\boldsymbol{x}\Big| + \Big|\int_\Omega \lambda^{-1}\boldsymbol{p}_k\cdot\boldsymbol{q}_k\,d\boldsymbol{x}\Big| \\ \le{}& \|\boldsymbol{y}_k\|_{L^2(\Omega)}\|\boldsymbol{v}_k\|_{L^2(\Omega)} + (\nu\lambda^{-1/4}\nabla\boldsymbol{p}_k,\lambda^{1/4}\nabla\boldsymbol{v}_k)_{L^2(\Omega)} \\ &+ k\omega(\sigma\lambda^{-1/4}\boldsymbol{p}_k,\lambda^{1/4}\boldsymbol{v}_k^\perp)_{L^2(\Omega)} + (\nu\lambda^{1/4}\nabla\boldsymbol{y}_k,\lambda^{-1/4}\nabla\boldsymbol{q}_k)_{L^2(\Omega)} \\ &+ k\omega(\sigma\lambda^{1/4}\boldsymbol{y}_k,\lambda^{-1/4}\boldsymbol{q}_k^\perp)_{L^2(\Omega)} + \lambda^{-1}\|\boldsymbol{p}_k\|_{L^2(\Omega)}\|\boldsymbol{q}_k\|_{L^2(\Omega)}.\end{aligned}$$

Applying the Cauchy-Schwarz inequality again several times, we obtain

$$\begin{aligned}\big|\mathcal{B}((\boldsymbol{y}_k,\boldsymbol{p}_k),(\boldsymbol{v}_k,\boldsymbol{q}_k))\big| \le{}& \big(\|\boldsymbol{y}_k\|^2_{L^2(\Omega)} + \lambda^{-1/2}(\nu\nabla\boldsymbol{p}_k,\nabla\boldsymbol{p}_k)_{L^2(\Omega)} \\ &+ k\omega\lambda^{-1/2}(\sigma\boldsymbol{p}_k,\boldsymbol{p}_k)_{L^2(\Omega)} + \lambda^{1/2}(\nu\nabla\boldsymbol{y}_k,\nabla\boldsymbol{y}_k)_{L^2(\Omega)} \\ &+ k\omega\lambda^{1/2}(\sigma\boldsymbol{y}_k,\boldsymbol{y}_k)_{L^2(\Omega)} + \lambda^{-1}\|\boldsymbol{p}_k\|^2_{L^2(\Omega)}\big)^{1/2} \\ &\big(\|\boldsymbol{v}_k\|^2_{L^2(\Omega)} + \lambda^{1/2}(\nu\nabla\boldsymbol{v}_k,\nabla\boldsymbol{v}_k)_{L^2(\Omega)} \\ &+ k\omega\lambda^{1/2}(\sigma\boldsymbol{v}_k,\boldsymbol{v}_k)_{L^2(\Omega)} + \lambda^{-1/2}(\nu\nabla\boldsymbol{q}_k,\nabla\boldsymbol{q}_k)_{L^2(\Omega)} \\ &+ k\omega\lambda^{-1/2}(\sigma\boldsymbol{q}_k,\boldsymbol{q}_k)_{L^2(\Omega)} + \lambda^{-1}\|\boldsymbol{q}_k\|^2_{L^2(\Omega)}\big)^{1/2} \\ ={}& \big(\|\boldsymbol{y}_k\|^2_{\mathbb{V}} + \lambda^{-1}\|\boldsymbol{p}_k\|^2_{\mathbb{V}}\big)^{1/2}\big(\|\boldsymbol{v}_k\|^2_{\mathbb{V}} + \lambda^{-1}\|\boldsymbol{q}_k\|^2_{\mathbb{V}}\big)^{1/2} \\ ={}& \|(\boldsymbol{y}_k,\boldsymbol{p}_k)\|_{\mathcal{P}}\|(\boldsymbol{v}_k,\boldsymbol{q}_k)\|_{\mathcal{P}}.\end{aligned}$$

Hence, we have proved the upper bound with $\overline{c}=1$. Now, we want to show the estimate from below. With the choice

$$(\boldsymbol{v}_k,\boldsymbol{q}_k) = \Big(\boldsymbol{y}_k - \frac{1}{\sqrt{\lambda}}\boldsymbol{p}_k - \frac{1}{\sqrt{\lambda}}\boldsymbol{p}_k^\perp, \boldsymbol{p}_k + \sqrt{\lambda}\boldsymbol{y}_k - \sqrt{\lambda}\boldsymbol{y}_k^\perp\Big),$$

we get the relations

$$\begin{aligned}
\mathcal{B}((\boldsymbol{y}_k,\boldsymbol{p}_k),(\boldsymbol{y}_k,\boldsymbol{p}_k)) =& \|\boldsymbol{y}_k\|^2_{L^2(\Omega)} + \lambda^{-1}\|\boldsymbol{p}_k\|^2_{L^2(\Omega)},\\
\mathcal{B}\Big((\boldsymbol{y}_k,\boldsymbol{p}_k),\Big(-\frac{1}{\sqrt{\lambda}}\boldsymbol{p}_k,\sqrt{\lambda}\boldsymbol{y}_k\Big)\Big) =& \sqrt{\lambda}(\nu\nabla\boldsymbol{y}_k,\nabla\boldsymbol{y}_k)_{L^2(\Omega)}\\
&+\frac{1}{\sqrt{\lambda}}(\nu\nabla\boldsymbol{p}_k,\nabla\boldsymbol{p}_k)_{L^2(\Omega)},\\
\mathcal{B}\Big((\boldsymbol{y}_k,\boldsymbol{p}_k),\Big(-\frac{1}{\sqrt{\lambda}}\boldsymbol{p}_k^\perp,-\sqrt{\lambda}\boldsymbol{y}_k^\perp\Big)\Big) =& k\omega\sqrt{\lambda}(\sigma\boldsymbol{y}_k,\boldsymbol{y}_k)_{L^2(\Omega)} + k\omega\frac{1}{\sqrt{\lambda}}(\sigma\boldsymbol{p}_k,\boldsymbol{p}_k)_{L^2(\Omega)}.
\end{aligned}$$

Altogether, with this choice, we obtain

$$\mathcal{B}((\boldsymbol{y}_k,\boldsymbol{p}_k),(\boldsymbol{v}_k,\boldsymbol{q}_k)) = \|\boldsymbol{y}_k\|^2_{\mathbb{V}} + \lambda^{-1}\|\boldsymbol{p}_k\|^2_{\mathbb{V}} = \|(\boldsymbol{y}_k,\boldsymbol{p}_k)\|^2_{\mathcal{P}}.$$

By using the fact that

$$\|(\boldsymbol{v}_k,\boldsymbol{q}_k)\|^2_{\mathcal{P}} = \Big\|\Big(\boldsymbol{y}_k - \frac{1}{\sqrt{\lambda}}\boldsymbol{p}_k - \frac{1}{\sqrt{\lambda}}\boldsymbol{p}_k^\perp, \boldsymbol{p}_k + \sqrt{\lambda}\boldsymbol{y}_k - \sqrt{\lambda}\boldsymbol{y}_k^\perp\Big)\Big\|^2_{\mathcal{P}} = 3\|(\boldsymbol{y}_k,\boldsymbol{p}_k)\|^2_{\mathcal{P}},$$

we arrive at the estimate of the supremum from below:

$$\sup_{0\neq(\boldsymbol{v}_k,\boldsymbol{q}_k)\in\mathbb{V}^2}\frac{\mathcal{B}((\boldsymbol{y}_k,\boldsymbol{p}_k),(\boldsymbol{v}_k,\boldsymbol{q}_k))}{\|(\boldsymbol{v}_k,\boldsymbol{q}_k)\|_{\mathcal{P}}} \geq \frac{1}{\sqrt{3}}\|(\boldsymbol{y}_k,\boldsymbol{p}_k)\|_{\mathcal{P}}.$$

Hence, we get $\underline{c} = 1/\sqrt{3}$. This completes the proof of the theorem. □

Due to the symmetry of the bilinear form $\mathcal{B}(.,.)$, inequalities (37) in Theorem 6 immediately yield existence and uniqueness of the solution of the variational problem (36).

Due to the supremum the discrete version of the left inequality in Theorem 6 (the so-called inf-sup condition) does in general not follow from the continuous version. However, in our case, we can repeat the proof step-by-step, and, finally, we arrive at the same inequalities in the discrete case where $\mathbb{V}^2$ is replaced by $\mathbb{V}_h^2$ with the same constants. Therefore, in matrix-vector notation, we have proved the inequalities

$$\underline{c}\,\|\underline{x}\|_{\mathcal{P}} \leq \sup_{\underline{z}\in\mathbb{R}^{4n}}\frac{(\mathcal{A}\underline{x},\underline{z})}{\|\underline{z}\|_{\mathcal{P}}} \leq \overline{c}\,\|\underline{x}\|_{\mathcal{P}} \quad \forall\underline{x}\in\mathbb{R}^{4n} \tag{38}$$

implying the condition number estimate

$$\kappa_{\mathcal{P}}(\mathcal{P}^{-1}\mathcal{A}) := \|\mathcal{P}^{-1}\mathcal{A}\|_{\mathcal{P}}\,\|\mathcal{A}^{-1}\mathcal{P}\|_{\mathcal{P}} \leq \overline{c}/\underline{c} = \sqrt{3}. \tag{39}$$

Theorem 3 yields now a robust convergence rate of the preconditioned MINRES method with

$$q = \frac{\kappa_{\mathcal{P}}(\mathcal{P}^{-1}\mathcal{A}) - 1}{\kappa_{\mathcal{P}}(\mathcal{P}^{-1}\mathcal{A}) + 1} \leq \frac{\sqrt{3}-1}{\sqrt{3}+1} \approx 0.267949.$$

Finally, we want to determine a preconditioner for the discretized system (23) in the case of $k = 0$. This is done in the same way as before. The matrix $\mathcal{A}$ is now given by

$$\mathcal{A} := \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}, \tag{40}$$

where $A := M_h$, $B := -K_h$ and $C := \lambda^{-1}M_h$. According to Theorem 4, we construct the two block diagonal preconditioners P_0 and P_1 with

$$P_0 = \begin{pmatrix} A & 0 \\ 0 & C + BA^{-1}B^T \end{pmatrix} \text{ and } P_1 = \begin{pmatrix} A + B^TC^{-1}B & 0 \\ 0 & C \end{pmatrix}.$$

By applying Theorem 5, we obtain the new preconditioner P with

$$P = \begin{pmatrix} [A,R]_{1/2} & 0 \\ 0 & [S,C]_{1/2} \end{pmatrix} = \begin{pmatrix} [A,R]_{1/2} & 0 \\ 0 & \lambda^{-1}[A,R]_{1/2} \end{pmatrix},$$

where

$$\begin{aligned}
[A,R]_{1/2} =& M_h^{1/2}(M_h^{-1/2}(M_h + \lambda K_h M_h^{-1} K_h) M_h^{-1/2})^{1/2} M_h^{1/2} \\
\sim& M_h^{1/2}(M_h^{-1/2} M_h M_h^{-1/2})^{1/2} M_h^{1/2} \\
&+ \sqrt{\lambda} M_h^{1/2}(M_h^{-1/2} K_h M_h^{-1} K_h M_h^{-1/2})^{1/2} M_h^{1/2} \\
=& M_h^{1/2} M_h^{-1/2} M_h^{1/2} M_h^{1/2} + \sqrt{\lambda} M_h^{1/2} M_h^{-1/2} K_h M_h^{-1/2} M_h^{1/2} \\
=& M_h + \sqrt{\lambda} K_h := D.
\end{aligned}$$

Hence, the preconditioner is given by

$$P = \begin{pmatrix} D & 0 \\ 0 & \lambda^{-1} D \end{pmatrix}. \tag{41}$$

We can again establish similar inequalities as in Theorem 6. Indeed, let us define the bilinear form

$$\mathcal{B}((y_0^c, p_0^c), (v_0^c, q_0^c)) := \int_\Omega y_0^c v_0^c - \nu \nabla p_0^c \cdot \nabla v_0^c + \nu \nabla y_0^c \cdot \nabla q_0^c + \lambda^{-1} p_0^c q_0^c \, d\boldsymbol{x}, \tag{42}$$

hence the variational problem (20) reads now as follows: Find $(y_0^c, p_0^c) \in V \times V$ with $V = H_0^1(\Omega)$ such that

$$\mathcal{B}((y_0^c, p_0^c), (v_0^c, q_0^c)) = \int_\Omega y_{d0}^c \cdot v_0^c \, d\boldsymbol{x} \tag{43}$$

for all test functions $(v_0^c, q_0^c) \in V \times V$. Moreover, defining the inner product

$$\begin{aligned}
((y,p),(v,q))_P^2 =& (y,v)_{L^2(\Omega)}^2 + \sqrt{\lambda}(\nu \nabla y, \nabla v)_{L^2(\Omega)} \\
&+ \lambda^{-1}((p,q)_{L^2(\Omega)}^2 + \sqrt{\lambda}(\nu \nabla p, \nabla q)_{L^2(\Omega)})
\end{aligned}$$

with associated norm

$$\|(y,p)\|_P^2 = \|y\|_{L^2(\Omega)}^2 + \sqrt{\lambda}(\nu \nabla y, \nabla y)_{L^2(\Omega)} + \lambda^{-1}(\|p\|_{L^2(\Omega)}^2 + \sqrt{\lambda}(\nu \nabla p, \nabla p)_{L^2(\Omega)}),$$

we can again show the following inequalities

$$\underline{c}\, \|(y_0^c, p_0^c)\|_P \le \sup_{0 \ne (v_0^c, q_0^c) \in \mathbb{V}} \frac{\mathcal{B}((y_0^c, p_0^c), (v_0^c, q_0^c))}{\|(v_0^c, q_0^c)\|_P} \le \overline{c}\, \|(y_0^c, p_0^c)\|_P \tag{44}$$

for all $(y_0^c, p_0^c) \in \mathbb{V}$ with constants $\underline{c}$ and $\overline{c}$ independent of all involved parameters. The upper bound of the supremum with the constant $\overline{c} = 1$ is again obtained by applying the triangle and the Cauchy-Schwarz inequalities. The estimate from below follows by the choice

$$(v_0^c, q_0^c) = \Big(y_0^c - \frac{1}{\sqrt{\lambda}} p_0^c, p_0^c + \sqrt{\lambda} y_0^c \Big).$$

For this choice, we obtain

$$\|(v_0^c, q_0^c)\|_P^2 = 2 \|(y_0^c, p_0^c)\|_P^2$$

and

$$\mathcal{B}((y_0^c, p_0^c), (v_0^c, q_0^c)) = \|(y_0^c, p_0^c)\|_P^2.$$

Hence, the constant for the lower bound is $\underline{c} = 1/\sqrt{2}$. The same arguments as above lead us to the estimate

$$\kappa_P(P^{-1}\mathcal{A}) \le \sqrt{2}, \tag{45}$$

which provides a robust convergence rate of the preconditioned MINRES method by Theorem 3 with

$$q \leq \frac{\sqrt{2}-1}{\sqrt{2}+1} \approx 0.171573.$$

Moreover, we obtain the existence of a unique solution of our optimality system.

Theorem 7. *The variational problem (17) of the optimality system has a unique solution.*

Proof. Since the functions y_d, y and p have a multiharmonic representation and since existence and uniqueness of a solution for the problems corresponding to the modes $k = 0, 1, 2, \ldots, N$ is guaranteed, we obtain existence and uniqueness of the whole optimality system (17) as well. Moreover, the multiharmonic representations of the state y and the co-state p given by (18) can be uniquely determined by inserting all solutions of the variational problems (36) for $k = 1, 2, \ldots, N$ and (43) for $k = 0$, which are the Fourier coefficients of y and p, into (18). $\square$

Altogether, for every mode $k = 0, 1, 2, \ldots, N$, we have determined a preconditioner such that the corresponding system can be solved by the preconditioned MINRES method with a robust convergence rate.

5.3. **Practical Implementation.** In practical applications, the diagonal blocks $D = \sqrt{\lambda} K_h + k\omega\sqrt{\lambda} M_{h,\sigma} + M_h$ of the preconditioner $\mathcal{P}$ in (34) of the discretized problem (21) for the case $k = 1, 2, \ldots, N$ and the diagonal blocks $D = M_h + \sqrt{\lambda} K_h$ of the preconditioner P in (41) of the problem (23) for the case $k = 0$ have to be replaced by diagonal blocks $\tilde{D}$, which are spectral equivalent to D, robust, symmetric positive definite and more cost efficient. The construction of such blocks can be done by techniques as for example multigrid methods or domain decomposition methods, see Mardal and Winther [26]. The practical block diagonal preconditioner for the case $k = 1, 2, \ldots, N$ is now given by

$$\tilde{\mathcal{P}} = \begin{pmatrix} \tilde{D} & 0 & 0 & 0 \\ 0 & \tilde{D} & 0 & 0 \\ 0 & 0 & \lambda^{-1}\tilde{D} & 0 \\ 0 & 0 & 0 & \lambda^{-1}\tilde{D} \end{pmatrix}. \tag{46}$$

The spectral equivalence of the diagonal blocks $\tilde{D} \sim D$ implies the spectral equivalence of the preconditioners $\tilde{\mathcal{P}} \sim \mathcal{P}$ with the same parameter independent constants $\underline{c}_D$ and $\overline{c}_D$. Hence, the condition number can be estimated by

$$\kappa_{\tilde{\mathcal{P}}}(\tilde{\mathcal{P}}^{-1}\mathcal{A}) \leq \kappa_{\mathcal{P}}(\mathcal{P}^{-1}\mathcal{A})\,(\overline{c}_D/\underline{c}_D) \tag{47}$$

where $\kappa_{\mathcal{P}}(\mathcal{P}^{-1}\mathcal{A}) \leq \sqrt{3}$ and $\underline{c}_D\tilde{D} \leq D \leq \overline{c}_D\tilde{D}$. In the case $k = 0$, we construct the practical preconditioner $\tilde{P}$ similarly, and the condition number of the preconditioned system can be again estimated by $\kappa_{\tilde{P}}(\tilde{P}^{-1}\mathcal{A}) \leq \kappa_P(P^{-1}\mathcal{A})(\overline{c}/\underline{c}) \leq \sqrt{2}(\overline{c}_D/\underline{c}_D)$. So, the new practical block diagonal preconditioners $\tilde{\mathcal{P}}$ yield again parameter independent convergence rates.

6. Discretization Error Analysis

In this section, we want to analyze the discretization errors $y - y_h$ and $p - p_h$ coming from the finite element discretization. The multiharmonic representation of y, y_h, p and p_h have been already introduced in Section 4. The analysis starts with proving that the discretization error of the Fourier coefficients can be estimated by the best approximation error. Afterwards we estimate the best approximation error by the interpolation error provided the Fourier coefficients are sufficiently smooth. Finally, we present an estimate for the complete discretization error $((y,p)-(y_h,p_h))$ in an appropriate norm.

Our decoupled variational problems for $k=1,...,N$ are given by: Find $(\boldsymbol{y}_k,\boldsymbol{p}_k)\in\mathbb{V}^2$ such that

$$\begin{aligned}\mathcal{B}((\boldsymbol{y}_k,\boldsymbol{p}_k),(\boldsymbol{v}_k,\boldsymbol{q}_k)) &= \int_\Omega \boldsymbol{y_d}_k\cdot\boldsymbol{v}_k\,d\boldsymbol{x}\\ &= \int_\Omega(\boldsymbol{y_d}_k,0)\cdot(\boldsymbol{v}_k,\boldsymbol{q}_k)\,d\boldsymbol{x}\\ &=: \langle F,(\boldsymbol{v}_k,\boldsymbol{q}_k)\rangle\end{aligned}$$

for all test functions $(\boldsymbol{v}_k,\boldsymbol{q}_k)\in\mathbb{V}^2$. Note that the error analysis for the case $k=0$ can be done analogously. Due to the assumption that the desired state $y_d(\boldsymbol{x},t)$ has a multiharmonic representation of the form (7), the corresponding discrete problems are given by: Find $(\boldsymbol{y}_{kh},\boldsymbol{p}_{kh})\in\mathbb{V}_h^2$ such that

$$\mathcal{B}((\boldsymbol{y}_{kh},\boldsymbol{p}_{kh}),(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh}))=\langle F,(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\rangle$$

for all test functions $(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\in\mathbb{V}_h^2$, which is equivalent to solving the linear system (21). Since $\mathbb{V}_h\subset\mathbb{V}$, we have the Galerkin orthogonality

$$\mathcal{B}((\boldsymbol{y}_k,\boldsymbol{p}_k),(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh}))-\mathcal{B}((\boldsymbol{y}_{kh},\boldsymbol{p}_{kh}),(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh}))=0,\quad\forall(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\in\mathbb{V}_h^2. \tag{48}$$

Due to linearity, the discretization error between the unknown solution (y,p) and its finite element approximation (y_h,p_h) can be deduced from the discretization error between the unknown Fourier coefficients and their finite element approximations.

Theorem 8. *Under the assumption that $(\boldsymbol{y}_k,\boldsymbol{p}_k)\in(H^2(\Omega))^4$ the discretization error for the Fourier coefficients can be estimated by*

$$\|(\boldsymbol{y}_k,\boldsymbol{p}_k)-(\boldsymbol{y}_{kh},\boldsymbol{p}_{kh})\|_{\mathcal{P}}\le c\,c_{par}(\lambda,k,\omega,\overline{\nu},\overline{\sigma},h)\,h\,|(\boldsymbol{y}_k,\boldsymbol{p}_k)|_{H^2(\Omega)}, \tag{49}$$

where c is a constant depending on the constants from the approximation theorem, $c^2_{par}(\lambda,k,\omega,\overline{\nu},\overline{\sigma},h)=\sqrt{\lambda}\overline{\nu}+(1+k\omega\sqrt{\lambda}\overline{\sigma})h^2$, and $|\cdot|_{H^2(\Omega)}$ is a weighted $H^2(\Omega)$-seminorm defined by the relation $|(\boldsymbol{y}_k,\boldsymbol{p}_k)|^2_{H^2(\Omega)}=|\boldsymbol{y}_k|^2_{H^2(\Omega)}+\lambda^{-1}|\boldsymbol{p}_k|^2_{H^2(\Omega)}$.

Proof. Inserting an arbitrary $(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\in\mathbb{V}_h^2$, using triangle inequality and the discrete inf-sup condition of Babuška-Aziz together with (48), we obtain the following estimate

$$\begin{aligned}&\|(\boldsymbol{y}_k,\boldsymbol{p}_k)-(\boldsymbol{y}_{kh},\boldsymbol{p}_{kh})\|_{\mathcal{P}}\le\|(\boldsymbol{y}_k,\boldsymbol{p}_k)-(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\|_{\mathcal{P}}+\|(\boldsymbol{y}_{kh},\boldsymbol{p}_{kh})-(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\|_{\mathcal{P}}\\
&\le\|(\boldsymbol{y}_k,\boldsymbol{p}_k)-(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\|_{\mathcal{P}}+\sqrt{3}\sup_{0\neq(\tilde{\boldsymbol{v}}_{kh},\tilde{\boldsymbol{q}}_{kh})\in\mathbb{V}_h^2}\frac{\mathcal{B}((\boldsymbol{y}_{kh},\boldsymbol{p}_{kh})-(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh}),(\tilde{\boldsymbol{v}}_{kh},\tilde{\boldsymbol{q}}_{kh}))}{\|(\tilde{\boldsymbol{v}}_{kh},\tilde{\boldsymbol{q}}_{kh})\|_{\mathcal{P}}}\\
&\le\|(\boldsymbol{y}_k,\boldsymbol{p}_k)-(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\|_{\mathcal{P}}+\sqrt{3}\underbrace{\sup_{0\neq(\tilde{\boldsymbol{v}}_{kh},\tilde{\boldsymbol{q}}_{kh})\in\mathbb{V}_h^2}\frac{\mathcal{B}((\boldsymbol{y}_{kh},\boldsymbol{p}_{kh})-(\boldsymbol{y}_k,\boldsymbol{p}_k),(\tilde{\boldsymbol{v}}_{kh},\tilde{\boldsymbol{q}}_{kh}))}{\|(\tilde{\boldsymbol{v}}_{kh},\tilde{\boldsymbol{q}}_{kh})\|_{\mathcal{P}}}}_{=0}\\
&+\sqrt{3}\sup_{0\neq(\tilde{\boldsymbol{v}}_{kh},\tilde{\boldsymbol{q}}_{kh})\in\mathbb{V}_h^2}\frac{\mathcal{B}((\boldsymbol{y}_k,\boldsymbol{p}_k)-(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh}),(\tilde{\boldsymbol{v}}_{kh},\tilde{\boldsymbol{q}}_{kh}))}{\|(\tilde{\boldsymbol{v}}_{kh},\tilde{\boldsymbol{q}}_{kh})\|_{\mathcal{P}}}\\
&\le\|(\boldsymbol{y}_k,\boldsymbol{p}_k)-(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\|_{\mathcal{P}}+\sqrt{3}\cdot 1\|(\boldsymbol{y}_k,\boldsymbol{p}_k)-(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\|_{\mathcal{P}}\\
&\le\underbrace{(1+\sqrt{3})}_{=:c_a}\|(\boldsymbol{y}_k,\boldsymbol{p}_k)-(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\|_{\mathcal{P}}.\end{aligned}$$

So we can estimate the discretization error by the best approximation error, i.e.

$$\|(\boldsymbol{y}_k,\boldsymbol{p}_k)-(\boldsymbol{y}_{kh},\boldsymbol{p}_{kh})\|_{\mathcal{P}}\le c_a\inf_{(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\in\mathbb{V}_h^2}\|(\boldsymbol{y}_k,\boldsymbol{p}_k)-(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\|_{\mathcal{P}}. \tag{50}$$

Thus, the best approximation error can be estimated by the interpolation error, i.e.

$$\inf_{(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\in\mathbb{V}_h^2} \|(\boldsymbol{y}_k,\boldsymbol{p}_k)-(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\|_{\mathcal{P}} \le \|(\boldsymbol{y}_k,\boldsymbol{p}_k)-I_h^2(\boldsymbol{y}_k,\boldsymbol{p}_k)\|_{\mathcal{P}}, \tag{51}$$

where $I_h^2 : \mathbb{V}^2 \to \mathbb{V}_h^2$ (respectively $I_h : \mathbb{V} \to \mathbb{V}_h$) is some interpolation operator. The $\mathcal{P}$-norm

$$\|(\boldsymbol{y}_k,\boldsymbol{p}_k)\|_{\mathcal{P}}^2 = \|\boldsymbol{y}_k\|_{\mathbb{V}}^2 + \lambda^{-1}\|\boldsymbol{p}_k\|_{\mathbb{V}}^2$$

with

$$\begin{aligned}\|\boldsymbol{y}_k\|_{\mathbb{V}}^2 &= \sqrt{\lambda}(\nu\nabla\boldsymbol{y}_k,\nabla\boldsymbol{y}_k)_{L^2(\Omega)} + k\omega\sqrt{\lambda}(\sigma\boldsymbol{y}_k,\boldsymbol{y}_k)_{L^2(\Omega)} + \|\boldsymbol{y}_k\|_{L^2(\Omega)}^2\\ &\le \sqrt{\lambda}\overline{\nu}|\boldsymbol{y}_k|_{H^1(\Omega)}^2 + (1+k\omega\sqrt{\lambda}\overline{\sigma})\|\boldsymbol{y}_k\|_{L^2(\Omega)}^2\end{aligned}$$

is bounded by

$$\begin{aligned}\|(\boldsymbol{y}_k,\boldsymbol{p}_k)\|_{\mathcal{P}}^2 \le{}& \sqrt{\lambda}\overline{\nu}|\boldsymbol{y}_k|_{H^1(\Omega)}^2 + (1+k\omega\sqrt{\lambda}\overline{\sigma})\|\boldsymbol{y}_k\|_{L^2(\Omega)}^2\\ &+ \lambda^{-1}\big(\sqrt{\lambda}\overline{\nu}|\boldsymbol{p}_k|_{H^1(\Omega)}^2 + (1+k\omega\sqrt{\lambda}\overline{\sigma})\|\boldsymbol{p}_k\|_{L^2(\Omega)}^2\big).\end{aligned}$$

Under the assumption that the Fourier coefficients are from $H^2(\Omega)$, the interpolation error can be estimated by

$$\begin{aligned}\|(\boldsymbol{y}_k,\boldsymbol{p}_k) - I_h^2(\boldsymbol{y}_k,\boldsymbol{p}_k)\|_{\mathcal{P}}^2 &= \|(I-I_h)^2(\boldsymbol{y}_k,\boldsymbol{p}_k)\|_{\mathcal{P}}^2\\ &= \|(I-I_h)\boldsymbol{y}_k\|_{\mathbb{V}}^2 + \lambda^{-1}\|(I-I_h)\boldsymbol{p}_k\|_{\mathbb{V}}^2\\ &\le \sqrt{\lambda}\overline{\nu}|(I-I_h)\boldsymbol{y}_k|_{H^1(\Omega)}^2 + (1+k\omega\sqrt{\lambda}\overline{\sigma})\|(I-I_h)\boldsymbol{y}_k\|_{L^2(\Omega)}^2\\ &+ \lambda^{-1}\big(\sqrt{\lambda}\overline{\nu}|(I-I_h)\boldsymbol{p}_k|_{H^1(\Omega)}^2 + (1+k\omega\sqrt{\lambda}\overline{\sigma})\|(I-I_h)\boldsymbol{p}_k\|_{L^2(\Omega)}^2\big)\\ &\le c_b^2\big(\sqrt{\lambda}\overline{\nu}h^2|\boldsymbol{y}_k|_{H^2(\Omega)}^2 + (1+k\omega\sqrt{\lambda}\overline{\sigma})h^4|\boldsymbol{y}_k|_{H^2(\Omega)}^2\big)\\ &+ \lambda^{-1}c_b^2\big(\sqrt{\lambda}\overline{\nu}h^2|\boldsymbol{p}_k|_{H^2(\Omega)}^2 + (1+k\omega\sqrt{\lambda}\overline{\sigma})h^4|\boldsymbol{p}_k|_{H^2(\Omega)}^2\big)\\ &= c_b^2\big((\sqrt{\lambda}\overline{\nu} + (1+k\omega\sqrt{\lambda}\overline{\sigma})h^2)h^2|\boldsymbol{y}_k|_{H^2(\Omega)}^2\\ &+ \lambda^{-1}\underbrace{(\sqrt{\lambda}\overline{\nu} + (1+k\omega\sqrt{\lambda}\overline{\sigma})h^2)}_{=:c_{par}^2(\lambda,k,\omega,\overline{\nu},\overline{\sigma},h)}h^2|\boldsymbol{p}_k|_{H^2(\Omega)}^2\big)\\ &= c_b^2 c_{par}^2(\lambda,k,\omega,\overline{\nu},\overline{\sigma},h)h^2(|\boldsymbol{y}_k|_{H^2(\Omega)}^2 + \lambda^{-1}|\boldsymbol{p}_k|_{H^2(\Omega)}^2),\end{aligned}$$

where c_b is a generic constant coming from applying the approximation theorem from finite element discretization theory, see Ciarlet [9]. So

$$\|(\boldsymbol{y}_k,\boldsymbol{p}_k) - I_h^2(\boldsymbol{y}_k,\boldsymbol{p}_k)\|_{\mathcal{P}} \le c_b c_{par}(\lambda,k,\omega,\overline{\nu},\overline{\sigma},h)h\underbrace{(|\boldsymbol{y}_k|_{H^2(\Omega)}^2 + \lambda^{-1}|\boldsymbol{p}_k|_{H^2(\Omega)}^2)^{1/2}}_{=:|(\boldsymbol{y}_k,\boldsymbol{p}_k)|_{H^2(\Omega)}},$$

where $|(\boldsymbol{y}_k,\boldsymbol{p}_k)|_{H^2(\Omega)}$ is a weighted vector-valued H^2-seminorm. Altogether the discretization error for the Fourier coefficients can be estimated by

$$\begin{aligned}\|(\boldsymbol{y}_k,\boldsymbol{p}_k) - (\boldsymbol{y}_{kh},\boldsymbol{p}_{kh})\|_{\mathcal{P}} &\le c_a \inf_{(\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\in\mathbb{V}_h^2}\|(\boldsymbol{y}_k,\boldsymbol{p}_k) - (\boldsymbol{v}_{kh},\boldsymbol{q}_{kh})\|_{\mathcal{P}}\\ &\le c_a\|(\boldsymbol{y}_k,\boldsymbol{p}_k) - I_h^2(\boldsymbol{y}_k,\boldsymbol{p}_k)\|_{\mathcal{P}}\\ &\le \underbrace{c_a c_b}_{=:c}\, c_{par}(\lambda,k,\omega,\overline{\nu},\overline{\sigma},h)\, h\, |(\boldsymbol{y}_k,\boldsymbol{p}_k)|_{H^2(\Omega)},\end{aligned} \tag{52}$$

where c is a constant consisting of c_a and of a constant depending on the constants from the approximation theorem. □

Now, we are in the position to estimate the complete discretization error $((y,p)-(y_h,p_h))$.

n_{iter}		ω								
		10^{-8}	10^{-6}	10^{-4}	10^{-2}	1	10^{2}	10^{4}	10^{6}	10^{8}
	10^{-8}	15	15	15	15	15	15	14	8	4
	10^{-6}	14	14	14	14	14	16	14	8	4
	10^{-4}	14	14	14	14	14	20	16	8	4
	10^{-2}	10	10	10	10	10	16	16	8	4
λ	1	6	6	6	6	8	14	16	8	4
	10^{2}	6	6	6	6	6	14	16	8	4
	10^{4}	4	4	4	4	6	14	16	8	4
	10^{6}	4	4	4	4	6	14	16	8	4
	10^{8}	4	4	4	4	6	14	16	8	4

TABLE 1. Number of MINRES iterations n_{iter} for different values of ω and λ on a 64×64 grid

Theorem 9. *Under the assumptions of Theorem 8, the complete discretization error $((y,p)-(y_h,p_h))$ can be estimated as follows:*

$$\|(y,p)-(y_h,p_h)\|_{\mathcal{P}} \leq c\, c_{par}(\lambda, N, \omega, \overline{\nu}, \overline{\sigma}, h)\, h\, |(y,p)|_{H^2(\Omega)}, \tag{53}$$

where

$$\|(y,p)\|_{\mathcal{P}}^2 = T\|(y_0^c, p_0^c)\|_P^2 + \frac{T}{2}\sum_{k=1}^{N} \|(\boldsymbol{y}_k, \boldsymbol{p}_k)\|_{\mathcal{P}}^2,$$

$$|(y,p)|_{H^2(\Omega)}^2 = T|(y_0^c, p_0^c)|_{H^2(\Omega)}^2 + \frac{T}{2}\sum_{k=1}^{N} |(\boldsymbol{y}_k, \boldsymbol{p}_k)|_{H^2(\Omega)}^2$$

are defined in the Fourier space. The seminorm $|\cdot|_{H^2(\Omega)}$ for the Fourier coefficients is given in Theorem 8.

The proof of estimate (53) immediately follows from Theorem 8. We mention that similar estimates can be obtained by space interpolation if y and p have only reduced regularity. More precisely, if y and p only belong to $H^{1+s}(\Omega)$ with some $s \in (0,1)$ the convergence rate reduces from h to h^s.

7. Numerical Results

We present here the first numerical results for a 2-dimensional example where we want to study the convergence behavior of our preconditioned MINRES solver. Let us consider our optimal control problem (4) - (5) with an harmonic desired state, given by the formula

$$y_d(\boldsymbol{x}, t) = 2(4 - x_1 x_2(x_2-1) + x_1^2 x_2(x_2-1))(\cos(\omega t) + \sin(\omega t)),$$

in the unit square domain $\Omega = (0,1)\times(0,1)$ that is unifomely discretized by triangles. In our computations, we used the standard continuous, piecewise linear finite element spaces. In order to study the robustness of our preconditioner, we have performed numerical experiments for several parameter settings. In particular, we have varied the values of the parameters ω and λ, whereas σ and ν were set to 1 and $h = 1/64$. Table 1 presents the number of MINRES iterations, which are needed to reduce the residual by a factor of 10^{-6}. The theoretical bound for that lies at 22 iterations. Hence, the numerical results enhance the theoretical ones having a preconditioned MINRES solver which provides a parameter independent convergence rate.

8. Conclusions and Outlook

We have considered a distributed multiharmonic parabolic optimal control problem, where the optimality system decouples into smaller systems for the single mode Fourier coefficients. This decoupling simplifies the analysis and the construction of fast and robust solvers. Indeed, the algebraic systems resulting from the finite element discretization can be solved by means of a preconditioned MINRES method. The main contribution of this paper is the construction of preconditioners which yield convergence rates that are independent of the discretization parameter and all other involved "bad" parameters. At the beginning, we assumed that the parameter σ is constant. However, in many practical applications, for example, in electromagnetics, this parameter, which corresponds to the conductivity, is only piecewise constant and vanishes in non-conducting materials. Hence, we have derived robust block diagonal preconditioners for both cases, where the simpler case of a constant conductivity gave us the right inspiration for the general case. Moreover, we have analyzed the error coming from the finite element discretization. Finally, we have presented some numerical results showing the robustness of our preconditioner.

In this paper we have assumed a multiharmonic setting of our optimal control problem. The more general time-periodic setting can be reduced to the multiharmonic case after time discretizion by means of truncating the Fourier series expansion. Therefore, the multiharmonic finite element discretization of time-periodic parabolic optimal control problems leads to the same systems of linear algebraic equation as in the multiharmonic case. Thus, the same preconditioned MINRES solvers can be used. However, the time discretizion requires a further error analysis that is connected with the convergence of the Fourier series. In the case of different observation and control regions, different observation norms and control inequality constraints imposed on the Fourier-coefficients, Kolmbauer and Kollmann constructed preconditioners that are robust with respect to all parameters with exception of the cost or regularization parameter [18]. The efficient treatment of inequality constraints for the control and the state is much harder. The inclusion of such inequality constraints into the cost functional as a penalty term is one technique to handle this problem. However, this makes the optimal controll problem non-linear. In the case of non-linear problems caused by the penalty technique or other non-linearities in the state equation, we lose the decoupling of the optimality system for the Fourier coefficients. However, the block-diagonal preconditioners constructed for the linear case should also work in some non-linear cases [7].

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