JOHANNES KEPLER UNIVERSITY LINZ

Institute of Computational Mathematics

Efficient Solvers for Multiharmonic Eddy Current Optimal Control Problems with Various Constraints

Michael Kolmbauer

Institute of Computational Mathematics, Johannes Kepler University
Altenberger Str. 69, 4040 Linz, Austria

NuMa-Report No. 2011-09 November 2011

A–4040 LINZ, Altenbergerstraße 69, Austria

Technical Reports before 1998:

1995

95-1	Hedwig Brandstetter *Was ist neu in Fortran 90?*	March 1995
95-2	G. Haase, B. Heise, M. Kuhn, U. Langer *Adaptive Domain Decomposition Methods for Finite and Boundary Element Equations.*	August 1995
95-3	Joachim Schöberl *An Automatic Mesh Generator Using Geometric Rules for Two and Three Space Dimensions.*	August 1995

1996

96-1	Ferdinand Kickinger *Automatic Mesh Generation for 3D Objects.*	February 1996
96-2	Mario Goppold, Gundolf Haase, Bodo Heise und Michael Kuhn *Preprocessing in BE/FE Domain Decomposition Methods.*	February 1996
96-3	Bodo Heise *A Mixed Variational Formulation for 3D Magnetostatics and its Finite Element Discretisation.*	February 1996
96-4	Bodo Heise und Michael Jung *Robust Parallel Newton-Multilevel Methods.*	February 1996
96-5	Ferdinand Kickinger *Algebraic Multigrid for Discrete Elliptic Second Order Problems.*	February 1996
96-6	Bodo Heise *A Mixed Variational Formulation for 3D Magnetostatics and its Finite Element Discretisation.*	May 1996
96-7	Michael Kuhn *Benchmarking for Boundary Element Methods.*	June 1996

1997

97-1	Bodo Heise, Michael Kuhn and Ulrich Langer *A Mixed Variational Formulation for 3D Magnetostatics in the Space* $H(rot) \cap H(div)$	February 1997
97-2	Joachim Schöberl *Robust Multigrid Preconditioning for Parameter Dependent Problems I: The Stokes-type Case.*	June 1997
97-3	Ferdinand Kickinger, Sergei V. Nepomnyaschikh, Ralf Pfau, Joachim Schöberl *Numerical Estimates of Inequalities in* $H^{\frac{1}{2}}$.	August 1997
97-4	Joachim Schöberl *Programmbeschreibung NAOMI 2D und Algebraic Multigrid.*	September 1997

From 1998 to 2008 technical reports were published by SFB013. Please see
`http://www.sfb013.uni-linz.ac.at/index.php?id=reports`
From 2004 on reports were also published by RICAM. Please see
`http://www.ricam.oeaw.ac.at/publications/list/`
For a complete list of NuMa reports see
`http://www.numa.uni-linz.ac.at/Publications/List/`

EFFICIENT SOLVERS FOR MULTIHARMONIC EDDY CURRENT OPTIMAL CONTROL PROBLEMS WITH VARIOUS CONSTRAINTS

MICHAEL KOLMBAUER

Abstract. This work is devoted to the multiharmonic analysis of eddy current optimal control problems in a time-periodic setting. In contrast to previous approaches, we derive the problem setting in a mixed variational formulation to enforce Coulomb gauge in an explicit way. Additionally, we include the cases of different control and observation domains, observation in certain energy spaces, observation at a final time and constraints to the control and the state. In all these cases we propose a new preconditioned MinRes solver for the frequency domain equations and show that this solver is robust with respect to the space and time discretization parameters. In some cases, we can also show robustness with respect to the involved model, cost and regularization parameters.

1. Introduction

This work is devoted to the development of efficient solution procedures for the following optimal control problem: Minimize the functional

$$
\begin{aligned}
J(\mathbf{y},\mathbf{u}) = &\frac{1}{2}\int_{\Omega_1\times(0,T)} |\mathbf{y}-\mathbf{y_d}|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t + \frac{\tau}{2}\int_{\Omega_1\times(0,T)} |\,\mathbf{curl}\,\mathbf{y}-\mathbf{y_c}|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&+ \frac{\alpha}{2}\int_{\Omega_1} |\mathbf{y}(T)-\mathbf{y_T}|^2 \mathrm{d}\mathbf{x} + \frac{\lambda}{2}\int_{\Omega_2\times(0,T)} |\mathbf{u}|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t,
\end{aligned} \tag{1}
$$

subject to the state equation

$$
\begin{cases}
\sigma \dfrac{\partial \mathbf{y}}{\partial t} + \mathbf{curl}(\nu\,\mathbf{curl}\,\mathbf{y}) = \mathbf{u}, & \text{in } \Omega\times(0,T),\\
\mathbf{y}\times\mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega\times(0,T),\\
\mathbf{y}(0) = \mathbf{y}(T), & \text{in } \Omega,
\end{cases} \tag{2}
$$

and possible constraints imposed on the Fourier coefficients of the control (see subsection 5.1) and of the state (see subsection 5.2).

Here Ω is a bounded, simply connected Lipschitz domain. Ω_1 and Ω_2 are non-empty subsets of Ω, i.e. $\Omega_1,\Omega_2 \subset \Omega \subset \mathbb{R}^3$. The reluctivity $\nu \in L^\infty(\Omega)$ and the conductivity $\sigma \in L^\infty(\Omega)$ are supposed to be uniformly positive, i.e.

$$0 < \nu_{\min} \le \nu(\mathbf{x}) \le \nu_{\max}, \quad \text{and} \quad 0 < \sigma_{\min} \le \sigma(\mathbf{x}) \le \sigma_{\max}, \qquad \forall \mathbf{x}\in\Omega.$$

In fact, we assume, that the reluctivity is independent of $|\,\mathbf{curl}\,\mathbf{y}|$, i.e. we assume that the eddy current problem (2) is linear. The regularization parameter $\lambda > 0$, the cost parameters $\tau \ge 0$, $\alpha \ge 0$ and $\mathbf{y_d}$, $\mathbf{y_c} \in L_2((0,T),\mathbf{L_2}(\Omega))$ and $\mathbf{y_T} \in \mathbf{L_2}(\Omega)$ are given data, where $\mathbf{y_d}$ represents the desired state, $\mathbf{y_c}$ represents the desired $\mathbf{curl}$ of the state and $\mathbf{y_T}$ represents the desired state at the final time T.

The author gratefully acknowledges the financial support by the Austrian Science Fund (FWF) under the grants P19255 and W1214-N15, project DK04. The author also thanks the Austria Center of Competence in Mechatronics (ACCM), which is a part of the COMET K2 program of the Austrian Government, for supporting his work on eddy current problems.

In order to derive a solution of this minimization problem, we derive the first order optimality conditions. We discretize in space and time in terms of a multiharmonic finite element method (MH-FEM). Indeed, this method is very useful for solving problems in time-periodic settings (see [19, 23] and the references therein).

The idea of using a multiharmonic approach as a non-standard discretization method in time and combining it with the finite element method is not new and has been used by many authors in different applications (e.g. [4, 10, 12, 28, 33]). Indeed in [5] and [19] Langer et al. provided a rigorous numerical analysis for the multiharmonic finite element method applied to the eddy current problem and the parabolic optimal control problem, respectively.

The fast solution of the corresponding large linear system of finite element frequency domain equations is crucial for the competitiveness of this method. Hence appropriate (parameter-robust) preconditioning is an important issue. We follow an abstract preconditioning strategy that has been rigorously analyzed in [35] and has already successfully applied to the multiharmonic finite element (MH-FEM) or boundary element method (MH-BEM) or their coupling (MH-FEM/BEM) in several works [8, 18, 19, 21, 22, 23, 20].

In [23] Kolmbauer and Langer analyze time-periodic eddy current optimal control problems with distributed control and construct a preconditioned MinRes solver, that yields parameter independent convergence rates. In fact,in [23] the applicability of their approach for time-periodic eddy current optimal control problems is shown with some eminent restrictions, that limits the applicability of their solver to general problems:

1) The observation domain and the control domain have to be equal the computational domain, i.e. $\Omega_1 = \Omega_2 = \Omega$.
2) The observation is restricted to be done in the L_2-norm, i.e. $\tau = 0$.
3) There is no control of the final time, i.e. $\alpha = 0$.
4) There are no control constraints involved.
5) There are no state constraints involved.

The construction of parameter robust solvers taking into account this modifications 1) - 5) is not straightforward, since we have to cope with various discretization and model parameters, that impinge upon the convergence rate of any iterative method. Anyhow, in all these cases an efficient preconditioned MinRes method can be constructed, that is robust at least in the space and time discretization parameters. In some cases we are able to prove even robustness with respect to more involved model parameters. In fact, the aim of these work is to generalize and extend the ideas derived in [18] for simple parabolic problems to eddy current problems. Therefore, we treat each issue 1) - 5) separately and provide the dependence of the convergence rate on the model parameters, if our solver is applied to the resulting system of equations. Furthermore, we explore the decoupling with respect to the modes of the Fourier coefficients in 1), 2), 4) and 5).

In some applications, it is reasonable to impose control constraints to the individual Fourier coefficients of the control $\mathbf{u}$. Therefore, we analyze this specific case.

Pointwise state constraints are of importance for instance to avoid undesired singularities in the optimal state. This issue can also be achieved, by adding pointwise state constraints to the Fourier coefficients of the state $\mathbf{y}$. Following [34] and [13], we use a Moreau Yosida regularization to incorporate these pointwise state constraints to the Fourier coefficients.

Since even linear inequality constraints give rise to nonlinear optimality systems, we apply a Newton-type approach for their solution, cf [13]. Every Newton step is a saddle point problem, that obtains structural similarities with the linear problems.

Therefore in these nonlinear cases, our focus is on the efficient solution of these linear systems arising at each Newton step.

It turns out, that in order to provide theoretical results, it is essential to have a Friedrichs-type inequality in some of the cases, especially in 1), 2) and 4). While the existence of a Friedrichs-type inequality in a H^1 setting only requires a non-empty Dirichlet boundary, in the $\mathbf{H}(\mathbf{curl})$ setting a Friedrichs-type inequality is only available for weakly divergence free functions. In fact this is a strong limitation, since we want to use the full $\mathbf{H}(\mathbf{curl})$ space in our computations. Therefore, we have to incorporate the weakly-divergence free condition in terms of additional Lagrange multipliers in a weak setting. Consequently, one part of this work addresses this issue.

The rest of this paper is organized as follows. In Section 2, we derive the optimality system of the model problem and discretize in space and time in terms of the multiharmonic finite element method. Section 3 is devoted to preconditioning of the resulting linear system of equations. After providing a theoretical basis according to a result by Zulehner [35], we apply the preconditioning technique to the model problem stated in Section 2. In Section 4, we generalize the results obtained in Section 3 to more general setting as different control and observation domains, and observation in some energy norm. In Section 5, we add control and state constraints to the Fourier coefficients of the control and state, respectively, and analyze the robustness of our proposed solver. The application of our solver to minimization problems with final time constraints is discussed in Section 6. We discuss the efficient implementation of our solver in Section 7. Finally, Section 8 draws some conclusions.

2. An optimal control problem with distributed control

In this section we add the additional equation

$$\mathbf{div}(\sigma \mathbf{y}) = 0, \quad \text{in } \Omega \times (0,T)$$

in the state equation (2) and construct a parameter robust and block diagonal preconditioner. Therefore, as a first model problem, we consider a problem with distributed control, i.e $\Omega = \Omega_1 = \Omega_2$, without any control or state constraints, for the special setting $\alpha = \tau = 0$. Therefore, we consider the model problem

$$\min_{\mathbf{y},\mathbf{u}} J(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_{\Omega\times(0,T)} |\mathbf{y}-\mathbf{y_d}|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t + \frac{\lambda}{2}\int_{\Omega\times(0,T)} |\mathbf{u}|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t, \tag{3}$$

subject to the state equation

$$\begin{cases} \sigma \dfrac{\partial \mathbf{y}}{\partial t} + \mathbf{curl}(\nu\,\mathbf{curl}\,\mathbf{y}) = \mathbf{u}, & \text{in } \Omega \times (0,T), \\ \mathbf{div}(\sigma \mathbf{y}) = 0, & \text{in } \Omega \times (0,T), \\ \mathbf{y} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega \times (0,T), \\ \mathbf{y}(0) = \mathbf{y}(T), & \text{in } \Omega. \end{cases} \tag{4}$$

The difference of the problem studied in [23] is the additional equation $\mathbf{div}(\sigma \mathbf{y}) = 0$.

2.1. **Optimality system.** In order to derive the necessary optimality conditions, we start by stating the appropriate spaces and deriving the space-time variational formulation of the state equation (4). Therefore we introduce the space of time-periodic $\mathbf{H}_0(\mathbf{curl})$ and H^1 functions with vanishing Dirichlet trace on the boundary

$\partial\Omega$:

$$\boldsymbol{\mathcal{H}_1} := \{\mathbf{y}(\cdot,t) \in \mathbf{H}(\mathbf{curl},\Omega) : \frac{\partial \mathbf{y}}{\partial t} \in \mathbf{L_2}(\Omega), \mathbf{y}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega, \mathbf{y}(\cdot,0) = \mathbf{y}(\cdot,T)\},$$
$$\mathcal{H}_2 := \{p(\cdot,t) \in H^1(\Omega) : p = 0 \text{ on } \partial\Omega, p(\cdot,0) = p(\cdot,T)\}.$$

Therefore, the space-time variational formulation of the state equation (4) is given by: Find $(\mathbf{y},\mu) \in \boldsymbol{\mathcal{H}_1} \times \mathcal{H}_2$, such that

(5)
$$\begin{cases} \displaystyle\int_{\Omega\times(0,T)} \left[\sigma\frac{\partial \mathbf{y}}{\partial t}\cdot\mathbf{p} + \nu\,\mathbf{curl}\,\mathbf{y}\cdot\mathbf{curl}\,\mathbf{p} + \sigma\,\mathbf{p}\cdot\nabla\mu\right]\,\mathrm{d}\mathbf{x}\,\mathrm{dt} = \int_{\Omega\times(0,T)} \mathbf{u}\cdot\mathbf{p}\,\mathrm{d}\mathbf{x}\,\mathrm{dt}, \\ \displaystyle\int_{\Omega\times(0,T)} \sigma\,\mathbf{y}\cdot\nabla\rho\,\mathrm{d}\mathbf{x}\,\mathrm{dt} = 0, \end{cases}$$

for all test functions $(\mathbf{p},\rho) \in \boldsymbol{\mathcal{H}_1}\times\mathcal{H}_2$. Due to the unique solvability of the state equation (5) (e.g. [2]), the minimization problem (3)-(4) has a unique solution (e.g. [31]).

In order to solve our minimization problem, we formulate the optimality system, also called Karush-Kuhn-Tucker system. Therefore, we formally consider the Lagrangian functional of (3) - (4), given by

$$\mathcal{L}(\mathbf{y},\mu,\mathbf{u},\mathbf{p},\rho) := J(\mathbf{y},\mathbf{u}) + \int_{\Omega\times(0,T)}\left[\sigma\,\frac{\partial\mathbf{y}}{\partial t}\cdot\mathbf{p} + \nu\,\mathbf{curl}\,\mathbf{y}\cdot\mathbf{curl}\,\mathbf{p} + \sigma\,\mathbf{p}\cdot\nabla\mu - \mathbf{u}\cdot\mathbf{p}\right]\mathrm{d}\mathbf{x}\,\mathrm{dt}$$
$$+ \int_{\Omega\times(0,T)} \sigma\,\mathbf{y}\cdot\nabla\rho\,\mathrm{d}\mathbf{x}\,\mathrm{dt}.$$

Here $\mathbf{p}$ and μ are the Lagrangian multipliers to the state $\mathbf{y}$ and ρ. Deriving the first order sufficient and necessary optimality conditions:

$$\text{Find } \mathbf{y},\mu,\mathbf{u},\mathbf{p},\rho : \quad \begin{cases} \nabla_{\mathbf{p}}\mathcal{L}(\mathbf{y},\mu,\mathbf{u},\mathbf{p},\rho) = \mathbf{0}, \\ \nabla_{\rho}\mathcal{L}(\mathbf{y},\mu,\mathbf{u},\mathbf{p},\rho) = \mathbf{0}, \\ \nabla_{\mathbf{y}}\mathcal{L}(\mathbf{y},\mu,\mathbf{u},\mathbf{p},\rho) = \mathbf{0}, \\ \nabla_{\mu}\mathcal{L}(\mathbf{y},\mu,\mathbf{u},\mathbf{p},\rho) = \mathbf{0}, \\ \nabla_{\mathbf{u}}\mathcal{L}(\mathbf{y},\mu,\mathbf{u},\mathbf{p},\rho) = \mathbf{0}, \end{cases}$$

we end up with the reduced optimality system in the weak form: Find $(\mathbf{y},\mathbf{p},\mu,\rho) \in \boldsymbol{\mathcal{H}_1}\times\boldsymbol{\mathcal{H}_1}\times\mathcal{H}_2\times\mathcal{H}_2$, such that

(6)
$$\begin{cases} -(\sigma\dfrac{\partial\mathbf{p}}{\partial t},\mathbf{v})_{Q_T} + (\nu\,\mathbf{curl}\,\mathbf{p},\mathbf{curl}\,\mathbf{v})_{Q_T} + (\mathbf{y},\mathbf{v})_{Q_T} + (\sigma\mathbf{v},\nabla\rho)_{Q_T} = (\mathbf{y_d},\mathbf{v})_{Q_T}, \\ (\sigma\mathbf{p},\nabla\eta)_{Q_T} = 0, \\ (\sigma\dfrac{\partial\mathbf{y}}{\partial t},\mathbf{q})_{Q_T} + (\nu\,\mathbf{curl}\,\mathbf{y},\mathbf{curl}\,\mathbf{q})_{Q_T} - \lambda^{-1}(\mathbf{p},\mathbf{q})_{Q_T} + (\sigma\mathbf{q},\nabla\mu)_{Q_T} = 0, \\ (\sigma\mathbf{y},\nabla\theta)_{Q_T} = 0, \end{cases}$$

for all test functions $(\mathbf{v},\mathbf{q},\eta,\theta) \in \boldsymbol{\mathcal{H}_1}\times\boldsymbol{\mathcal{H}_1}\times\mathcal{H}_2\times\mathcal{H}_2$. Note, that we have already incorporated that $\mathbf{u} = \lambda^{-1}\mathbf{p}$, arising from the last equation in the optimality system (6). Additionally, we use the abbreviation $Q_T = (0,T)\times\mathbf{L_2}(\Omega)$ and $(\cdot,\cdot)_{Q_T}$ for the corresponding inner product. This space-time variational formulation is the starting point of discretization in space and time.

2.2. **Multiharmonic discretization.** Let us assume, that the desired state $\mathbf{y_d}$ is multiharmonic, i.e. $\mathbf{y_d}$ has the form

$$\mathbf{y_d} = \sum_{k=0}^{N} \mathbf{y^c_{d,k}}\cos(k\omega t) + \mathbf{y^s_{d,k}}\sin(k\omega t), \tag{7}$$

where the Fourier coefficients are given by the formulas

$$\mathbf{y^c_{d,k}} = \frac{2}{T}\int_0^T \mathbf{y_d}\cos(k\omega t)\mathrm{d}t \quad \text{and} \quad \mathbf{y^s_{d,k}} = \frac{2}{T}\int_0^T \mathbf{y_d}\sin(k\omega t)\mathrm{d}t.$$

We mention that the multiharmonic representation (7) can also be seen as an approximation of a time-periodic desired state $\mathbf{y_d}$ by a truncated Fourier series. Due to the linearity of the optimality system (6) the state $\mathbf{y}$ and the co-state $\mathbf{p}$ are multiharmonic as well and therefore also have a representation in terms of a truncated Fourier series, i.e.

$$(8) \quad \mathbf{y} = \sum_{k=0}^{N} \mathbf{y^c_k}\cos(k\omega t) + \mathbf{y^s_k}\sin(k\omega t) \quad \text{and} \quad \mathbf{p} = \sum_{k=0}^{N} \mathbf{p^c_k}\cos(k\omega t) + \mathbf{p^s_k}\sin(k\omega t),$$

with unknown coefficients $(\mathbf{y^c_k}, \mathbf{y^s_k})$ and $(\mathbf{p^c_k}, \mathbf{p^s_k})$. Similarly, ρ and μ obtain representations in terms of truncated Fourier series, i.e.

$$\rho = \sum_{k=0}^{N} \rho^c_k\cos(k\omega t) + \rho^s_k\sin(k\omega t) \quad \text{and} \quad \mu = \sum_{k=0}^{N} \mu^c_k\cos(k\omega t) + \mu^s_k\sin(k\omega t),$$

with unknown coefficients (ρ^c_k, ρ^s_k) and (μ^c_k, μ^s_k).

Due to the linearity of the reduced optimality problem and the $L_2(0,T)$ orthogonality of the sine and cosine functions, the huge $(8N+4)\times(8N+4)$ system decouples into N 8×8 systems of partial differential equations for the two Fourier coefficients of each, the state $\mathbf{y}$ and the co-state $\mathbf{p}$ and the Lagrangian multipliers μ and ρ belonging to the mode k, and a 4×4 system of partial differential equations for the mode $k=0$. Note, that the mode $k=0$ has to be treated separately. Clearly we do not have to solve for $\mathbf{p^s_0}$, $\mathbf{y^s_0}$, μ^s_0 and ρ^s_0, since $\sin(0\omega t)=0$. Consequently, the problem that we deal with reads as follows: For each mode $k=0,1,\dots,N$, find the Fourier coefficients $(\mathbf{y^c_k}, \mathbf{y^s_k}, \mathbf{p^c_k}, \mathbf{p^s_k}) \in \mathbf{H_0(curl}, \Omega)^4$ and $(\mu^c_k, \mu^s_k, \rho^c_k, \rho^s_k) \in H^1_0(\Omega)^4$, such that

$$\begin{cases}
-\omega k(\sigma\mathbf{p^s_k}, \mathbf{v^c_k})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{p^c_k}, \mathbf{curl}\,\mathbf{v^c_k})_{\mathbf{L_2}(\Omega)} \\
\qquad +(\mathbf{y^c_k}, \mathbf{v^c_k})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma\mathbf{v^c_k}, \nabla\rho^c_k)_{\mathbf{L_2}(\Omega)} = (\mathbf{y^c_{d,k}}, \mathbf{v^c_k})_{\mathbf{L_2}(\Omega)}, \\
\qquad \omega k(\sigma\mathbf{p^c_k}, \nabla\eta^c_k)_{\mathbf{L_2}(\Omega)} = 0, \\
\omega k(\sigma\mathbf{p^c_k}, \mathbf{v^s_k})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{p^s_k}, \mathbf{curl}\,\mathbf{v^s_k})_{\mathbf{L_2}(\Omega)} \\
\qquad +(\mathbf{y^s_k}, \mathbf{v^s_k})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma\mathbf{v^s_k}, \nabla\rho^s_k)_{\mathbf{L_2}(\Omega)} = (\mathbf{y^s_{d,k}}, \mathbf{v^s_k})_{\mathbf{L_2}(\Omega)}, \\
\qquad \omega k(\sigma\mathbf{p^s_k}, \nabla\eta^s_k)_{\mathbf{L_2}(\Omega)} = 0, \\
\omega k(\sigma\mathbf{y^s_k}, \mathbf{q^c_k})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{y^c_k}, \mathbf{curl}\,\mathbf{q^c_k})_{\mathbf{L_2}(\Omega)} \\
\qquad -\lambda^{-1}(\mathbf{p^c_k}, \mathbf{q^c_k})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma\mathbf{q^c_k}, \nabla\mu^c_k)_{\mathbf{L_2}(\Omega)} = 0, \\
\qquad \omega k(\sigma\mathbf{y^c_k}, \nabla\theta^c_k)_{\mathbf{L_2}(\Omega)} = 0, \\
-\omega k(\sigma\mathbf{y^c_k}, \mathbf{q^s_k})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{y^s_k}, \mathbf{curl}\,\mathbf{q^s_k})_{\mathbf{L_2}(\Omega)} \\
\qquad -\lambda^{-1}(\mathbf{p^s_k}, \mathbf{q^s_k})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma\mathbf{q^s_k}, \nabla\mu^s_k)_{\mathbf{L_2}(\Omega)} = 0, \\
\qquad \omega k(\sigma\mathbf{y^s_k}, \nabla\theta^s_k)_{\mathbf{L_2}(\Omega)} = 0,
\end{cases}$$

for all test functions $(\mathbf{v^c_k}, \mathbf{v^s_k}, \mathbf{q^c_k}, \mathbf{q^s_k}) \in \mathbf{H_0(curl}, \Omega)^4$ and $(\eta^c_k, \eta^s_k, \theta^c_k, \theta^s_k) \in H^1_0(\Omega)^4$. Note, that the weakly divergence free conditions for the cosine and the sine or the state and co-state are scaled by $k\omega$ at each mode k. Since the whole problem decouples to a block-diagonal one corresponding to each mode k, for preconditioning purpose, it is enough to discuss the block for a fixed mode k. Therefore, for the rest of this section, we concentrate on solving a 8×8 block for a fixed mode k and therefore we omit the subindex k. For simplicity, we introduce the abbreviation

$$\Upsilon := (\mathbf{y^c}, \mathbf{y^s}, \mathbf{p^c}, \mathbf{p^s}) \quad \text{and} \quad \Psi := (\mu^c, \mu^s, \rho^c, \rho^s),$$
$$\Phi := (\mathbf{v^c}, \mathbf{v^s}, \mathbf{q^c}, \mathbf{q^s}) \quad \text{and} \quad \Theta := (\eta^c, \eta^s, \theta^c, \theta^s).$$

Therefore, we end up with the variational formulation: Find $(\Upsilon, \Psi) \in \mathbf{H}_0(\mathbf{curl}, \Omega)^4 \times H_0^1(\Omega)^4$, such that

$$\mathcal{A}((\Upsilon, \Psi), (\Phi, \Theta)) = \mathcal{F}((\Phi, \Theta)) \tag{9}$$

for all $(\Phi, \Theta) \in \mathbf{H}_0(\mathbf{curl}, \Omega)^4 \times H_0^1(\Omega)^4$, where the left-hand side $\mathcal{A}$ is given by

$$\mathcal{A}((\Upsilon, \Psi), (\Phi, \Theta)) := a(\Upsilon, \Phi) + b(\Phi, \Psi) + b(\Upsilon, \Theta),$$

with the bilinear forms

$$\begin{aligned} a(\Upsilon, \Phi) &:= a_a((\mathbf{y^c}, \mathbf{y^s}), (\mathbf{v^c}, \mathbf{v^s})) + a_b((\mathbf{v^c}, \mathbf{v^s}), (\mathbf{p^c}, \mathbf{p^s})) \\ &\quad + a_b((\mathbf{y^c}, \mathbf{y^s}), (\mathbf{q^c}, \mathbf{q^s})) - a_c((\mathbf{p^c}, \mathbf{p^s}), (\mathbf{q^c}, \mathbf{q^s})), \\ b(\Phi, \Psi) &:= \omega k \sum_{j \in \{c,s\}} \left[(\sigma \mathbf{v^j}, \nabla \rho^j)_{\mathbf{L_2}(\Omega)} + (\sigma \mathbf{q^j}, \nabla \mu^j)_{\mathbf{L_2}(\Omega)} \right], \end{aligned} \tag{10}$$

that are further decomposed into

$$\begin{aligned} a_a((\mathbf{y^c}, \mathbf{y^s}), (\mathbf{v^c}, \mathbf{v^s})) &:= \sum_{j \in \{c,s\}} (\mathbf{y^j}, \mathbf{v^j})_{\mathbf{L_2}(\Omega)}, \\ a_b((\mathbf{y^c}, \mathbf{y^s}), (\mathbf{q^c}, \mathbf{q^s})) &:= \omega k \left[(\sigma \mathbf{y^s}, \mathbf{q^c})_{\mathbf{L_2}(\Omega)} - (\sigma \mathbf{y^c}, \mathbf{q^s})_{\mathbf{L_2}(\Omega)} \right] \\ &\quad + \sum_{j \in \{c,s\}} (\nu \, \mathbf{curl} \, \mathbf{y^j}, \mathbf{curl} \, \mathbf{q^j})_{\mathbf{L_2}(\Omega)}, \\ a_c((\mathbf{p^c}, \mathbf{p^s}), (\mathbf{q^c}, \mathbf{q^s})) &:= \sum_{j \in \{c,s\}} \lambda^{-1} (\mathbf{p^j}, \mathbf{q^j})_{\mathbf{L_2}(\Omega)}. \end{aligned}$$

The right-hand side $\mathcal{F}$ is given by

$$\mathcal{F}((\Phi, \Theta)) = (\mathbf{y_d^c}, \mathbf{v^c})_{\mathbf{L_2}(\Omega)} + (\mathbf{y_d^s}, \mathbf{v^s})_{\mathbf{L_2}(\Omega)}.$$

Indeed, the bilinear form $\mathcal{A}$ is symmetric and indefinite. Well-posedness of the variational problem (9) will be shown in the next section, using the Theorem of Babuška-Aziz [3]. The variational formulation (9) is the starting point of the discretization in space.

2.3. **Discretization in space.** We use a quasi-uniform and shape-regular triangulation $\mathcal{T}_h$, with mesh size $h > 0$, of the computational domain Ω with tetrahedral elements. On this mesh, we consider Nédélec basis functions of lowest order yielding a conforming finite element subspace $\mathcal{ND}_0(\mathcal{T}_h)$ of $\mathbf{H}_0(\mathbf{curl})$. In order to approximate $H_0^1(\Omega)$ functions, we use continuous piecewise linear Lagrange finite elements resulting in the conforming finite element subspace $S_1(\mathcal{T}_h)$ of $H_0^1(\Omega)$. The finite element discretization of each 8×8 block leads to a 8×8 block-matrix $\mathcal{A}$ of the following form:

$$\underbrace{\left(\begin{array}{cccc|cccc} \mathbf{M} & \cdot & \mathbf{K}_\nu & -\mathbf{M}_{\sigma,\mathbf{k}\omega} & \cdot & \cdot & \mathbf{D}^T & \cdot \\ \cdot & \mathbf{M} & \mathbf{M}_{\sigma,\mathbf{k}\omega} & \mathbf{K}_\nu & \cdot & \cdot & \cdot & \mathbf{D}^T \\ \mathbf{K}_\nu & \mathbf{M}_{\sigma,\mathbf{k}\omega} & -\lambda^{-1}\mathbf{M} & \cdot & \mathbf{D}^T & \cdot & \cdot & \cdot \\ -\mathbf{M}_{\sigma,\mathbf{k}\omega} & \mathbf{K}_\nu & \cdot & -\lambda^{-1}\mathbf{M} & \cdot & \mathbf{D}^T & \cdot & \cdot \\ \hline \cdot & \cdot & \mathbf{D} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mathbf{D} & \cdot & \cdot & \cdot & \cdot \\ \mathbf{D} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \mathbf{D} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right)}_{=:\mathcal{A}} =: \begin{pmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \cdot \end{pmatrix} \tag{11}$$

Here the mass matrix $\mathbf{M}$, the conductivity matrix $\mathbf{M_{k\omega,\sigma}}$, the stiffness matrix $\mathbf{K_\nu}$ and the divergence matrix $\mathbf{D}$ arise from the finite element discretization of the corresponding bilinear forms

$$\mathbf{M} : (\cdot,\cdot)_{\mathbf{L_2}(\Omega)}, \quad \mathbf{M_{\sigma,k\omega}} : k\omega(\sigma\cdot,\cdot)_{\mathbf{L_2}(\Omega)}, \quad \mathbf{K_\nu} : (\nu\nabla\cdot,\nabla\cdot)_{\mathbf{L_2}(\Omega)}, \quad \mathbf{D} : (\sigma\cdot,\nabla\cdot)_{\mathbf{L_2}(\Omega)}.$$

The matrices fulfill the following properties: $\mathbf{M}$ and $\mathbf{M_{\sigma,k\omega}}$ are symmetric and positive definite. $\mathbf{K_\nu}$ is symmetric and positive semi-definite and due to the discrete inf-sup stability $\mathbf{D}$ has full rank. Hence, we have to solve a linear system of finite element equations of the form

$$\mathcal{A}\mathbf{w} = \mathbf{f}, \tag{12}$$

where the system matrix $\mathcal{A}$ is given by (11) and $\mathbf{f}$ is nothing but the finite element discretization of $\mathcal{F}$. In fact, the system matrix $\mathcal{A}$ is symmetric and indefinite and obtains a double saddle-point structure. Since $\mathcal{A}$ is symmetric, the system can be solved by a Minimal Residual (MinRes) method [27]. Typically, the convergence rate of any iterative method deteriorates with respect to the meshsize h and the "bad" parameters $k\omega$, ν, σ and λ, if applied to the unpreconditioned system (12). Therefore, preconditioning is a challenging topic.

3. Preconditioning

This section is devoted to the fast solution of (12). In [23], Kolmbauer and Langer propose a parameter-robust preconditioner for the multiharmonic finite element equations, derived from the time-periodic eddy current control problem. Indeed the latter mentioned solver corresponds to the block $\mathbf{A}$ in (11). Using this result, we again can construct a parameter-robust solver for the model problem (11).

Furthermore, the cases handled in this work are not capable with the preconditioning theory used in [19]. Therefore we use a generalized preconditioning result from Zulehner in [35], that is very useful for proving almost parameter-independent bounds in all these cases.

3.1. Abstract preconditioning theory.

In this subsection we briefly recall an abstract result of Zulehner [35], that extends the classical theory of Brezzi [7]. Let V and Q be Hilbert spaces with the inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_Q$. The associated norms are given by $\|\cdot\|_V = \sqrt{(\cdot,\cdot)_V}$ and $\|\cdot\|_Q = \sqrt{(\cdot,\cdot)_Q}$. Furthermore, let X be the product space $X = V \times Q$, equipped with the inner product $((v,q),(w,r))_X = (v,w)_V + (q,r)_Q$ and the associated norm $\|(v,q)\|_X = \sqrt{((v,q),(v,q))_X}$. Consider a mixed variational problem in the product space $X = V \times Q$: Find $z = (w,r) \in X$, such that

$$\mathcal{A}(z,y) = \mathcal{F}(y), \quad \text{for all } y \in X$$

with

$$\mathcal{A}(z,y) = a(w,v) + b(v,r) + b(w,q) - c(r,q), \quad \text{and} \quad \mathcal{F}(y) = f(v) + g(q)$$

for $y = (v,q)$ and $z = (w,r)$. We introduce the operator $B \in L(V,Q^*)$ and its adjoint $B^* \in L(Q,V^*)$ by the identities

$$\langle Bw,q\rangle = b(w,q) \quad \text{and} \quad \langle B^*r,v\rangle = \langle Bv,r\rangle,$$

respectively. Furthermore we denote by $\mathcal{A} \in L(X,X^*)$ the operator induced by

$$\langle \mathcal{A}x,y\rangle = \mathcal{A}(x,y).$$

The next theorem provides necessary and sufficient conditions for establishing parameter independent bounds for $\|\mathcal{A}x\|_{X^*}$ and can be found in Zulehner [35].

Theorem 1 ([35, Theorem 2.6]). *If there are constants $\underline{c}_w$, $\underline{c}_r$, $\overline{c}_w$, $\overline{c}_r > 0$, such that*

$$\underline{c}_w \|w\|_V^2 \leq a(w,w) + \|Bw\|_{Q^*}^2 \leq \overline{c}_w \|w\|_V^2, \quad \text{for all } w \in V \tag{13}$$

and

$$\underline{c}_r \|r\|_Q^2 \leq c(r,r) + \|B^* r\|_{V^*}^2 \leq \overline{c}_r \|r\|_Q^2, \quad \text{for all } r \in Q, \tag{14}$$

then

$$\underline{c}\|z\|_X \leq \|\mathcal{A}x\|_{X^*} \leq \overline{c}\|z\|_X, \quad \text{for all } z \in X \tag{15}$$

is satisfied with constants $\underline{c}$, $\overline{c} > 0$ that depend only on $\underline{c}_w$, $\overline{c}_w$, $\underline{c}_r$, $\overline{c}_r$.

As stated in [35, Remark 2], for the special case $c(\cdot,\cdot) = 0$, Theorem 1 simplifies to the classical Theorem of Brezzi. Additionally the successive result gives explicit bounds for $\underline{c}$ and $\overline{c}$.

Theorem 2 (Brezzi). *If $c(\cdot,\cdot) = 0$ and if there exist constants α_1, α_2, β_1 and $\beta_2 > 0$, such that*

- $|a(w,v)| \leq \alpha_2 \|w\|_V \|v\|_V, \forall w, v \in V,$
- $|b(v,r)| \leq \beta_2 \|v\|_V \|r\|_Q, \forall v \in V, \forall r \in Q,$
- *the inf-sup condition of $a(\cdot,\cdot)$ on kernel of $b(\cdot,\cdot)$ holds, i.e.*

  $$\inf_{v \in V_0} \sup_{w \in V_0} \frac{a(w,v)}{\|w\|_V \|v\|_V} \geq \alpha_1 \quad \text{and} \quad \inf_{w \in V_0} \sup_{v \in V_0} \frac{a(w,v)}{\|w\|_V \|v\|_V} \geq \alpha_1$$

  with $V_0 := \ker B = \{v \in V | b(v,r) = 0, \forall r \in Q\}$
- *and the inf-sup condition of $b(\cdot,\cdot)$ holds, i.e.*

  $$\inf_{r \in Q} \sup_{v \in V} \frac{b(v,r)}{\|v\|_V \|r\|_Q} \geq \beta_1,$$

then (15) is satisfied with the constants

$$\underline{c} = \frac{\alpha_1}{1 + \left(\frac{\alpha_2}{\beta_1}\right)^2} \quad \text{and} \quad \overline{c} = \frac{\alpha_2 + \sqrt{\alpha_2^2 + 4\beta_2^2}}{2}.$$

As exposed in [35], an immediate consequence of (15) is an estimate of the condition number $\kappa(\mathcal{A})$:

$$\kappa(\mathcal{A}) = \|\mathcal{A}\|_{L(X,X^*)} \|\mathcal{A}^{-1}\|_{L(X^*,X)} \leq \frac{\overline{c}}{\underline{c}}.$$

Therefore, robust estimates of the form (15), imply a robust estimate for the condition number. More precisely, (15) means, that solving the discrete variational problem connected with the inner product in X will supply a good preconditioner for $\mathcal{A}$.

3.2. **Preconditioning the MH-FEM matrices.** We use the preconditioning technique of the last section to analyze a preconditioner for the MH-FEM problem (11), resulting from the simple model problem (3)-(4). Later on, this block-diagonal preconditioner is reused and analyzed in the constrained settings.

In [23] Kolmbauer and Langer considered (3)-(4) without the divergence constraint. Indeed, the multiharmonic finite element discretization results in the left-upper block

$$\mathbf{A} := \begin{pmatrix} \mathbf{M} & & \mathbf{K} & \mathbf{M}_{\sigma,\mathbf{k}\omega} \\ & \mathbf{M} & -\mathbf{M}_{\sigma,\mathbf{k}\omega} & \mathbf{K} \\ \mathbf{K} & -\mathbf{M}_{\sigma,\mathbf{k}\omega} & -\lambda^{-1}\mathbf{M} & \\ \mathbf{M}_{\sigma,\mathbf{k}\omega} & \mathbf{K} & & -\lambda^{-1}\mathbf{M} \end{pmatrix}$$

of $\mathcal{A}$ in (11). They propose and analyze the block-diagonal preconditioner

$$\mathcal{C} := \operatorname{diag}\left(\sqrt{\lambda}\mathbf{E}, \sqrt{\lambda}\mathbf{E}, \frac{1}{\sqrt{\lambda}}\mathbf{E}, \frac{1}{\sqrt{\lambda}}\mathbf{E}\right), \tag{16}$$

where the block $\mathbf{E}$ is given by $\mathbf{E} := \mathbf{K}_{\boldsymbol{\nu}} + \mathbf{M}_{\boldsymbol{\sigma},\mathbf{k}\boldsymbol{\omega}} + \frac{1}{\sqrt{\lambda}}\mathbf{M}$. They show, that this choice leads to parameter-independent estimates in the inf-sup and sup-sup conditions, and consequently to a parameter-independent condition number estimate

$$\kappa_{\mathcal{C}}(\mathcal{C}^{-1}\mathbf{A}) := \|\mathcal{C}^{-1}\mathbf{A}\|_{\mathcal{C}}\|\mathbf{A}^{-1}\mathcal{C}\|_{\mathcal{C}} \leq \sqrt{3} \approx 1.73205.$$

In order to construct an appropriate preconditioner for $\mathcal{A}$, we use $\mathcal{C}$ to obtain a block-diagonal preconditioner for the full system matrix. Having in mind a Schur complement type preconditioner, we propose

$$\begin{aligned}\mathcal{P} &= \operatorname{diag}\left(\mathcal{C}, \mathbf{B}\mathcal{C}^{-1}\mathbf{B}^T\right)\\ &= \operatorname{diag}\left(\sqrt{\lambda}\mathbf{E}, \sqrt{\lambda}\mathbf{E}, \frac{1}{\sqrt{\lambda}}\mathbf{E}, \frac{1}{\sqrt{\lambda}}\mathbf{E},\right.\\ &\qquad \left.\frac{1}{\sqrt{\lambda}}\mathbf{D}\mathbf{E}^{-1}\mathbf{D}^T, \frac{1}{\sqrt{\lambda}}\mathbf{D}\mathbf{E}^{-1}\mathbf{D}^T, \sqrt{\lambda}\mathbf{D}\mathbf{E}^{-1}\mathbf{D}^T, \sqrt{\lambda}\mathbf{D}\mathbf{E}^{-1}\mathbf{D}^T\right).\end{aligned} \tag{17}$$

Since $\mathbf{D}$ has full rank, we have that $\mathbf{D}\mathbf{E}^{-1}\mathbf{D}^T$ is symmetric and positive definite. Therefore, $\mathcal{P}$ is symmetric and positive definite, which is essential if this preconditioner is used to accelerate the MinRes method. For convenience, we introduce the following abbreviation for the finite element functions

$$\Upsilon_h := (\mathbf{y_h^c}, \mathbf{y_h^s}, \mathbf{p_h^c}, \mathbf{p_h^s}) \quad \text{and} \quad \Psi_h := (\mu_h^c, \mu_h^s, \rho_h^c, \rho_h^s),$$
$$\Phi_h := (\mathbf{v_h^c}, \mathbf{v_h^s}, \mathbf{q_h^c}, \mathbf{q_h^s})^T \quad \text{and} \quad \Theta_h := (\eta_h^c, \eta_h^s, \theta_h^c, \theta_h^s).$$

In order to analyze the block-diagonal preconditioner $\mathcal{P}$, we switch back to the variational framework in the finite element spaces and introduce the non-standard norm $\|\cdot\|_{\mathcal{C}_1}$ in $\mathcal{ND}_0(\mathcal{T}_h)$:

$$\|\mathbf{y_h}\|_{\mathcal{C}_1}^2 = (\nu\,\mathbf{curl}\,\mathbf{y_h}, \mathbf{curl}\,\mathbf{y_h})_{\mathbf{L_2}(\Omega)} + k\omega(\sigma\mathbf{y_h}, \mathbf{y_h})_{\mathbf{L_2}(\Omega)} + \frac{1}{\sqrt{\lambda}}(\mathbf{y_h}, \mathbf{y_h})_{\mathbf{L_2}(\Omega)}.$$

This definition gives rise to non-standard norms in the product spaces

$$\|(\mathbf{y_h^c}, \mathbf{y_h^s})\|_{\mathcal{C}_1}^2 := \sqrt{\lambda}\sum_{j\in\{c,s\}}\|\mathbf{y_h^j}\|_{\mathcal{C}_1}^2 \quad \text{and} \quad \|(\mathbf{y_h^c}, \mathbf{y_h^s})\|_{\mathcal{C}_2}^2 := \frac{1}{\sqrt{\lambda}}\sum_{j\in\{c,s\}}\|\mathbf{y_h^j}\|_{\mathcal{C}_1}^2, \tag{18}$$

and furthermore

$$\|\Upsilon_h\|_{\mathcal{C}}^2 := \|(\mathbf{y_h^c}, \mathbf{y_h^s})\|_{\mathcal{C}_1}^2 + \|(\mathbf{p_h^c}, \mathbf{p_h^s})\|_{\mathcal{C}_2}^2. \tag{19}$$

Finally, we define a non-standard norm in the product space $\mathcal{ND}_0(\mathcal{T}_h)^4 \times \mathcal{S}_1(\mathcal{T}_h)^4$, by

$$\|(\Upsilon_h, \Psi_h)\|_{\mathcal{P}}^2 := \|\Upsilon_h\|_{\mathcal{C}}^2 + \left(\sup_{\Phi_h\in\mathcal{ND}_0(\mathcal{T}_h)^4}\frac{b(\Phi_h, \Psi_h)}{\|\Phi_h\|_{\mathcal{C}}}\right)^2.$$

We mention, that $\sup\frac{b(\Phi_h,\Psi_h)}{\|\Phi_h\|_{\mathcal{C}}}$ is really a norm in $H_0^1(\Omega)^4$. The main result is summarized in the following Lemma, that claims that an inf-sup condition and a sup-sup condition are fulfilled in the non-standard norm with constants $\frac{1}{2\sqrt{3}}$ and $\frac{1+\sqrt{5}}{2}$.

Lemma 1. *We have*

$$\frac{1}{2\sqrt{3}}\|(\Upsilon_h, \Psi_h)\|_{\mathcal{P}} \leq \sup_{(\Phi_h,\Theta_h)\neq 0}\frac{\mathcal{A}((\Upsilon_h, \Psi_h), (\Phi_h, \Theta_h))}{\|(\Phi_h, \Theta_h)\|_{\mathcal{P}}} \leq \frac{1+\sqrt{5}}{2}\|(\Upsilon_h, \Psi_h)\|_{\mathcal{P}}$$

for all $(\Upsilon_h, \Psi_h) \in \mathcal{ND}_0(\mathcal{T}_h)^4 \times \mathcal{S}_1(\mathcal{T}_h)^4$.

Proof. From [23, Theorem 2], we obtain that the bilinear form $a(\cdot,\cdot)$ is bounded with constant 1 and satisfies an inf-sup condition on the kernel of $b(\cdot,\cdot)$ with constant $1/\sqrt{3}$. Boundedness of $b(\cdot,\cdot)$ easily follows with constant 1. Finally, by definition of $\|\cdot\|_{\bar{P}}$, the bilinear form $b(\cdot,\cdot)$ satisfies an inf-sup condition with constant 1. Consequently the lower and upper bound follow from Theorem 2. $\square$

Hence it follows by the theorem of Babuška-Aziz, that there exists a unique solution of the corresponding variational problem (9), and that the solution continuously depends on the data, uniformly in h, k, ω, σ, ν and λ. Hence we conclude, that the block-diagonal preconditioner $\mathcal{P}$ yields robust MinRes convergence rates with respect to the space discretization parameter h and the time discretization parameters k and ω, as well as to the model parameters σ, ν and λ. Additionally, from Lemma 1, we immediately obtain that the spectral condition number of the preconditioned system can be estimated by a constant, i.e.

$$\kappa_{\mathcal{P}}(\mathcal{P}^{-1}\mathcal{A}) := \|\mathcal{P}^{-1}\mathcal{A}\|_{\mathcal{P}}\|\mathcal{A}^{-1}\mathcal{P}\|_{\mathcal{P}} \leq \sqrt{3}(1+\sqrt{5}) \approx 5.60503.$$

4. Modifications in the minimization functional

In this Section we extend the basic model problem (3)-(4) step by step to more involving problems. In all the cases discussed in the introduction, the resulting system matrices obtain high structural similarities to $\mathcal{A}$ as in (11). It turns out, that we have to pay a price in the sense, that we lose robustness with respect to specific model parameters in order to get robustness with respect to modifications in the cost functional and/or the state equation. The idea is to use the same block-diagonal preconditioner $\mathcal{P}$, given in (17), as for the optimal control problem with distributed control. The main ingredients of the proofs in this section is a weighted Friedrichs-type inequality in $\mathbf{H}_0(\mathbf{curl})$. Therefore, the following Lemma plays a crucial role and claims, that in the discrete space of weakly divergence-free $\mathbf{H}_0(\mathbf{curl})$ functions, the mass matrix is basically dominated by the stiffness matrix, with constants independent of any involved discretization and model parameters.

Lemma 2. *For any* $\mathbf{v_h} \in \mathcal{ND}_0(\mathcal{T}_h)$ *which satisfies*

$$(\sigma \mathbf{v_h}, \nabla q_h)_{\mathbf{L_2}(\Omega)} = 0, \quad \forall q_h \in \mathcal{S}_1(\mathcal{T}_h),$$

we have

$$\|\sigma^{1/2}\mathbf{v_h}\|_{\mathbf{L_2}(\Omega)} \leq C\|\sigma^{1/2}\,\mathbf{curl}\,\mathbf{v_h}\|_{\mathbf{L_2}(\Omega)},$$

where C *is independent of the mesh size* h *and the related parameter* σ.

Proof. See [17, Lemma 4.2]. $\square$

In the following, we use Lemma 2 in a very specific style.

Corollary 1. *For any* $\mathbf{v_h} \in \mathcal{ND}_0(\mathcal{T}_h)$ *which satisfies*

$$(\sigma \mathbf{v_h}, \nabla q_h)_{\mathbf{L_2}(\Omega)} = 0, \quad \forall q_h \in \mathcal{S}_1(\mathcal{T}_h),$$

we have

$$\|\mathbf{v_h}\|_{L_2(\Omega)} \leq C_{\sigma,\nu}\|\tilde{\nu}^{1/2}\,\mathbf{curl}\,\mathbf{v_h}\|_{\mathbf{L_2}(\Omega)},$$

where $C_{\sigma,\nu} = C\sqrt{\frac{\sigma_{\max}}{\sigma_{\min}\nu_{\min}}}$.

Proof. Using Lemma 2, we obtain

$$\begin{aligned}\|\mathbf{v_h}\|^2_{\mathbf{L_2}(\Omega)} &= (\mathbf{v_h},\mathbf{v_h})_{\mathbf{L_2}(\Omega)} \le \frac{1}{\sigma_{\min}}(\sigma\mathbf{v_h},\mathbf{v_h})_{\mathbf{L_2}(\Omega)} \le \frac{C^2}{\sigma_{\min}}(\sigma\,\mathbf{curl}\,\mathbf{v_h},\mathbf{curl}\,\mathbf{v_h})_{\mathbf{L_2}(\Omega)}\\ &\le \frac{C^2\sigma_{\max}}{\sigma_{\min}}(\mathbf{curl}\,\mathbf{v_h},\mathbf{curl}\,\mathbf{v_h})_{\mathbf{L_2}(\Omega)} \le \frac{C^2\sigma_{\max}}{\sigma_{\min}\nu_{\min}}(\nu\,\mathbf{curl}\,\mathbf{v_h},\mathbf{curl}\,\mathbf{v_h})_{\mathbf{L_2}(\Omega)}\\ &= C^2_{\sigma,\nu}\|\nu^{1/2}\,\mathbf{curl}\,\mathbf{v_h}\|^2_{\mathbf{L_2}(\Omega)}.\end{aligned}$$

□

Note, that for constant conductivity, i.e. $\sigma = const$, $C_{\sigma,\nu}$ does not depend on σ. Corollary 1 allows us, to derive a very specific inf-sup and sup-sup bound for $a_b(\cdot,\cdot)$.

Lemma 3. *We have*

$$\underline{c}_b\|(\mathbf{y^c_h},\mathbf{y^s_h})\|_{\mathcal{C}_1} \le \sup_{(\mathbf{y^c_h},\mathbf{y^s_h})\in\mathcal{ND}_0(\mathcal{T}_h)^2}\frac{a_b((\mathbf{y^c_h},\mathbf{y^s_h}),(\mathbf{q^c_h},\mathbf{q^s_h}))}{\|(\mathbf{q^c_h},\mathbf{q^s_h})\|_{\mathcal{C}_2}} \le \overline{c}_b\|(\mathbf{y^c_h},\mathbf{y^s_h})\|_{\mathcal{C}_1}$$

and

$$\underline{c}_b\|(\mathbf{y^c_h},\mathbf{y^s_h})\|_{\mathcal{C}_2} \le \sup_{(\mathbf{q^c_h},\mathbf{q^s_h})\in\mathcal{ND}_0(\mathcal{T}_h)^2}\frac{a_b((\mathbf{q^c_h},\mathbf{q^s_h}),(\mathbf{y^c_h},\mathbf{y^s_h}))}{\|(\mathbf{q^c_h},\mathbf{q^s_h})\|_{\mathcal{C}_1}} \le \overline{c}_b\|(\mathbf{y^c_h},\mathbf{y^s_h})\|_{\mathcal{C}_2}$$

for all $(\mathbf{y^c_h},\mathbf{y^s_h}) \in \mathcal{ND}_0(\mathcal{T}_h)^2$ *fulfilling* $(\sigma\mathbf{y^c_h},\nabla p_h) = 0$ *and* $(\sigma\mathbf{y^s_h},\nabla p_h) = 0$ *for all* $p_h \in \mathcal{S}_1(\mathcal{T}_h)$. *Here* $\underline{c}_b$ *and* $\overline{c}_b$ *are constants that depend on* λ, σ *and* ν, *but are independent of* h, N *and* ω.

Proof. First we show boundedness of $a_b(\cdot,\cdot)$, indeed

$$a_b((\mathbf{y^c_h},\mathbf{y^s_h}),(\mathbf{q^c_h},\mathbf{q^s_h})) \le \|(\mathbf{y^c_h},\mathbf{y^s_h})\|_{\mathcal{C}_1}\|(\mathbf{q^c_h},\mathbf{q^s_h})\|_{\mathcal{C}_2}.$$

In order to verify the inf-sup condition of $a_b(\cdot,\cdot)$, we use the special choice $(\mathbf{q^c_h},\mathbf{q^s_h}) = (\mathbf{y^c_h}+\mathbf{y^s_h},\mathbf{y^s_h}-\mathbf{y^c_h})$ for the test functions. Due to the restriction on $\mathbf{y^c_h}$ and $\mathbf{y^c_h}$ to be weakly divergence free, we can use Corollary 1:

$$\begin{aligned}&a_b((\mathbf{y^c_h},\mathbf{y^s_h}),(\mathbf{y^c_h},\mathbf{y^s_h})) + a_b((\mathbf{y^c_h},\mathbf{y^s_h}),(\mathbf{y^s_h},-\mathbf{y^c_h})) \ge\\ &\quad\ge \sum_{j\in\{c,s\}}\Big[(\nu\,\mathbf{curl}\,\mathbf{y^j_h},\mathbf{curl}\,\mathbf{y^j_h})_{\mathbf{L_2}(\Omega)} + k\omega(\sigma\mathbf{y^j_h},\mathbf{y^j_h})_{\mathbf{L_2}(\Omega)}\Big]\\ &\quad\ge \sum_{j\in\{c,s\}}\Big[\frac{1}{2}(\nu\,\mathbf{curl}\,\mathbf{u^j_h},\mathbf{curl}\,\mathbf{u^j_h})_{\mathbf{L_2}(\Omega)} + \frac{1}{2C_{\sigma,\nu}}(\mathbf{y^j_h},\mathbf{y^j_h})_{\mathbf{L_2}(\Omega)} + k\omega(\sigma\mathbf{y^j_h},\mathbf{y^j_h})_{\mathbf{L_2}(\Omega)}\Big]\\ &\quad\ge c\|(\mathbf{y^c_h},\mathbf{y^s_h})\|^2_{\mathcal{C}_2}.\end{aligned}$$

Note, that for this special choice we have $\|(\mathbf{v^c_h},\mathbf{v^s_h})\|_{\mathcal{C}_2} = \sqrt{2}\|(\mathbf{y^c_h},\mathbf{y^s_h})\|_{\mathcal{C}_2}$. Therefore the first result follows. Since $a_b(\cdot,\cdot)$ is skew symmetric, the same estimate can be obtained for the adjoint setting. □

4.1. **Different control and observation domains.** In many practical applications, it makes no sense to locate the observation and/or control in the full computational domain Ω. Therefore we assume that the observation and control domains Ω_1 and Ω_2 are simply connected subdomains of the computational domain Ω, i.e. $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$. In order to deal with the different support of the observation and control, we define the prolongation operators $\mathbf{P_i}$, $i = 1, 2$, by

$$\begin{aligned}&\mathbf{P_i} : \mathbf{L_2}(\Omega_i) \to \mathbf{L_2}(\Omega),\\ &(\mathbf{P_i u},\mathbf{v})_{\mathbf{L_2}(\Omega)} = (\mathbf{u},\mathbf{v})_{\mathbf{L_2}(\Omega_i)}, \quad \forall\mathbf{u}\in\mathbf{L_2}(\Omega_i)\ \forall\mathbf{v}\in\mathbf{L_2}(\Omega),\end{aligned}$$

and the appropriate restriction operators $\mathbf{P_i^*}$, $i = 1, 2$, by

$$\mathbf{P_i^*} : \mathbf{L_2}(\Omega) \to \mathbf{L_2}(\Omega_i),$$
$$(\mathbf{P_i u}, \mathbf{v})_{L_2(\Omega)} = (\mathbf{u}, \mathbf{P_i^* v})_{\mathbf{L_2}(\Omega_i)}, \quad \forall \mathbf{u} \in \mathbf{L_2}(\Omega_i)\ \forall \mathbf{v} \in \mathbf{L_2}(\Omega).$$

Again we use the scaling argument and obtain the following minimization problem:

$$\min_{\mathbf{y},\mathbf{u}} J(\mathbf{y}, \mathbf{u}) = \frac{1}{2} \int_{\Omega_1 \times (0,T)} |\mathbf{P_1}^*[\mathbf{y} - \mathbf{y_d}]|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{dt} + \frac{\lambda}{2} \int_{\Omega_2 \times (0,T)} |\mathbf{u}|^2 \mathrm{d}\mathbf{x} \, \mathrm{dt}, \tag{20}$$

subject to the state equation

$$\begin{cases} \sigma \dfrac{\partial \mathbf{y}}{\partial t} + \mathbf{curl}(\nu \ \mathbf{curl}\, \mathbf{y}) = \mathbf{P_2 u}, & \text{in } \Omega \times (0, T), \\ \mathbf{div}(\sigma \mathbf{y}) = 0, & \text{in } \Omega \times (0, T), \\ \mathbf{y} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega \times (0, T), \\ \mathbf{y}(0) = \mathbf{y}(T), & \text{on } \Omega \times \{0\}. \end{cases} \tag{21}$$

In this setting we have

$$\mathcal{A}_1((\Upsilon, \Psi), (\Phi, \Theta)) := a^1(\Upsilon, \Phi) + b(\Phi, \Psi) + b(\Upsilon, \Theta)$$

with

$$a^1((\mathbf{y^c}, \mathbf{y^s}, \mathbf{p^c}, \mathbf{p^s}), (\mathbf{v^c}, \mathbf{v^s}, \mathbf{q^c}, \mathbf{q^s})) := a_a^1((\mathbf{y^c}, \mathbf{y^s}), (\mathbf{v^c}, \mathbf{v^s}))$$
$$+ a_b((\mathbf{v^c}, \mathbf{v^s}), (\mathbf{p^c}, \mathbf{p^s})) + a_b((\mathbf{y^c}, \mathbf{y^s}), (\mathbf{q^c}, \mathbf{q^s})) - a_c^1((\mathbf{p^c}, \mathbf{p^s}), (\mathbf{q^c}, \mathbf{q^s}))$$

and the new bilinear forms

$$a_a^1((\mathbf{y^c}, \mathbf{y^s}), (\mathbf{v^c}, \mathbf{v^s})) := \sum_{j \in \{c,s\}} (\mathbf{y^j}, \mathbf{v^j})_{\mathbf{L_2}(\Omega_1)},$$
$$a_c^1((\mathbf{p^c}, \mathbf{p^s}), (\mathbf{q^c}, \mathbf{q^s})) := \sum_{j \in \{c,s\}} \lambda^{-1} (\mathbf{p^j}, \mathbf{q^j})_{\mathbf{L_2}(\Omega_2)}.$$

The main result is summarized in the following Lemma, that claims that an inf-sup condition and a sup-sup condition are fulfilled, where the constants are independent of h, N and ω.

Lemma 4. *We have*

$$\underline{c}_1 \|(\Upsilon_h, \Psi_h)\|_{\mathcal{P}} \le \sup_{(\Phi_h, \Theta_h) \neq \mathbf{0}} \frac{\mathcal{A}_1((\Upsilon_h, \Psi_h), (\Phi_h, \Theta_h))}{\|(\Phi_h, \Theta_h)\|_{\mathcal{P}}} \le \overline{c}_1 \|(\Upsilon_h, \Psi_h)\|_{\mathcal{P}}$$

for all $(\Upsilon_h, \Psi_h) \in \mathcal{ND}_0(\mathcal{T}_h)^4 \times \mathcal{S}_1(\mathcal{T}_h)^4$. *Here the constants* $\underline{c}_1$, $\overline{c}_1$ *are independent of* h, k *and* ω.

Proof. In order to proof boundedness and the inf-sup condition of $a(\cdot, \cdot)$ on the kernel of $b(\cdot, \cdot)$, we use Theorem 1. We have to verify the lower and upper bounds for $a_a^1(\cdot, \cdot)$ and $a_c^1(\cdot, \cdot)$. Indeed, we have

$$0 \le a_a^1((\mathbf{y_h^c}, \mathbf{y_h^s}), (\mathbf{y_h^c}, \mathbf{y_h^s})) \le a_a((\mathbf{y_h^c}, \mathbf{y_h^s}), (\mathbf{y_h^c}, \mathbf{y_h^s})) \le \|(\mathbf{y_h^c}, \mathbf{y_h^s})\|_{\mathcal{C}_1}^2$$

and

$$0 \le a_c^1((\mathbf{p_h^c}, \mathbf{p_h^s}), (\mathbf{p_h^c}, \mathbf{p_h^s})) \le a_c((\mathbf{p_h^c}, \mathbf{p_h^s}), (\mathbf{p_h^c}, \mathbf{p_h^s})) \le \|(\mathbf{p_h^c}, \mathbf{p_h^s})\|_{\mathcal{C}_2}^2.$$

Furthermore we use the result of Lemma 3 to obtain the inf-sup conditions for $a_b(\cdot,\cdot)$. Therefore, combining the last estimates yields

$$
\begin{aligned}
c\|(\mathbf{y_h^c},\mathbf{y_h^s})\|_{\mathcal{C}_2}^2 &\leq a_a((\mathbf{y_h^c},\mathbf{y_h^s}),(\mathbf{y_h^c},\mathbf{y_h^s})) + \left(\sup_{0\neq(\mathbf{q_h^c},\mathbf{q_h^s})} \frac{a_b((\mathbf{y_h^c},\mathbf{y_h^s}),(\mathbf{q_h^c},\mathbf{q_h^s}))}{\|(\mathbf{q_h^c},\mathbf{q_h^s})\|_{\mathcal{C}_2}}\right)^2 \\
&\leq \|(\mathbf{y_h^c},\mathbf{y_h^s})\|_{\mathcal{C}_2}^2, \\
c\|(\mathbf{p_h^c},\mathbf{p_h^s})\|_{\mathcal{C}_2}^2 &\leq a_c((\mathbf{p_h^c},\mathbf{p_h^s}),(\mathbf{p_h^c},\mathbf{p_h^s})) + \left(\sup_{0\neq(\mathbf{v_h^c},\mathbf{v_h^s})} \frac{a_b((\mathbf{v_h^c},\mathbf{v_h^s}),(\mathbf{p_h^c},\mathbf{p_h^s}))}{\|(\mathbf{v_h^c},\mathbf{v_h^s})\|_{\mathcal{C}_2}}\right)^2 \\
&\leq \|(\mathbf{p_h^c},\mathbf{p_h^s})\|_{\mathcal{C}_2}^2,
\end{aligned}
$$

for all $(\mathbf{y_h^c},\mathbf{y_h^s},\mathbf{p_h^c},\mathbf{p_h^s})$ in the kernel of $b(\cdot,\cdot)$, where c is some generic constant, independent of h, ω and k. Now with Theorem 1, we obtain that the inf-sup condition of $a(\cdot,\cdot)$ on the kernel of $b(\cdot,\cdot)$ is satisfied. Boundedness of $a(\cdot,\cdot)$ follows easily with constant 1. Furthermore boundedness and the inf-sup condition of $b(\cdot,\cdot)$ can be done analogously to Lemma 1. Therefore, by Theorem 2, the desired result follows. □

4.2. **Observation in the energy norm.** Instead of the vector potential $\mathbf{y}$, we want to observe the magnetic flux density $\mathbf{B} = \mathbf{curl}\,\mathbf{y}$. Therefore we obtain the following equivalent minimization problem:

$$
\min_{\mathbf{y},\mathbf{u}} J(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_{\Omega\times(0,T)} |\mathbf{curl}\,\mathbf{y} - \mathbf{y_c}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \frac{\lambda}{2}\int_{\Omega\times(0,T)} |\mathbf{u}|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t, \tag{22}
$$

subject to the state equation

$$
\begin{cases}
\sigma\dfrac{\partial \mathbf{y}}{\partial t} + \mathbf{curl}(\nu\ \mathbf{curl}\,\mathbf{y}) = \mathbf{u}, & \text{in } \Omega\times(0,T), \\
\mathbf{div}(\sigma\mathbf{y}) = 0, & \text{in } \Omega\times(0,T), \\
\mathbf{y}\times\mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega\times(0,T), \\
\mathbf{y}(0) = \mathbf{y}(T), & \text{on } \Omega\times\{0\}.
\end{cases} \tag{23}
$$

In this setting we have

$$
\mathcal{A}_2((\Upsilon,\Psi),(\Phi,\Theta)) := a^2(\Upsilon,\Phi) + b(\Phi,\Psi) + b(\Upsilon,\Theta)
$$

with

$$
\begin{aligned}
a^2((\mathbf{y^c},\mathbf{y^s},\mathbf{p^c},\mathbf{p^s}),(\mathbf{v^c},\mathbf{v^s},\mathbf{q^c},\mathbf{q^s})) &:= a_a^2((\mathbf{y^c},\mathbf{y^s}),(\mathbf{v^c},\mathbf{v^s})) \\
&+ a_b((\mathbf{v^c},\mathbf{v^s}),(\mathbf{p^c},\mathbf{p^s})) + a_b((\mathbf{y^c},\mathbf{y^s}),(\mathbf{q^c},\mathbf{q^s})) - a_c((\mathbf{p^c},\mathbf{p^s}),(\mathbf{q^c},\mathbf{q^s}))
\end{aligned}
$$

and the new bilinear form

$$
a_a^2((\mathbf{y^c},\mathbf{y^s}),(\mathbf{v^c},\mathbf{v^s})) := \sum_{j\in\{c,s\}} (\mathbf{curl}\,\mathbf{y^j},\mathbf{curl}\,\mathbf{v^j})_{\mathbf{L_2}(\Omega)}.
$$

The main result is summarized in the following Lemma, that claims that the inf-sup condition in the Theorem of Babuška-Aziz is fulfilled in the non-standard norm.

Lemma 5. *We have*

$$
\underline{c}_2\|(\Upsilon_h,\Psi_h)\|_{\mathcal{P}} \leq \sup_{(\Phi_h,\Theta_h)\neq\mathbf{0}} \frac{\mathcal{A}_2((\Upsilon_h,\Psi_h),(\Phi_h,\Theta_h))}{\|(\Phi_h,\Theta_h)\|_{\mathcal{P}}} \leq \overline{c}_2\|(\Upsilon_h,\Psi_h)\|_{\mathcal{P}}
$$

for all $(\Upsilon_h,\Psi_h)\in\mathcal{ND}_0(\mathcal{T}_h)^4\times\mathcal{S}_1(\mathcal{T}_h)^4$. *Here the constants* $\underline{c}_2$, $\overline{c}_2$ *are independent of* h, k *and* ω.

Proof. The proof is basically the same as the proof of Lemma 4. The main differences are the lower and upper bounds for $a_a^2(\cdot,\cdot)$ and $a_c^2(\cdot,\cdot)$. We have

$$0 \leq a_a^2((\mathbf{y_h^c}, \mathbf{y_h^s}), (\mathbf{y_h^c}, \mathbf{y_h^s})) \leq \frac{1}{\nu_{\min}} \|(\mathbf{y_h^c}, \mathbf{y_h^s})\|_{\mathcal{C}_2}^2$$

and

$$0 \leq a_c^2((\mathbf{p_h^c}, \mathbf{p_h^s}), (\mathbf{p_h^c}, \mathbf{p_h^s})) \leq \|(\mathbf{p_h^c}, \mathbf{p_h^s})\|_{\mathcal{C}_2}^2.$$

This completes the proof. □

5. Control and the state constraints

5.1. **Control constraints on the Fourier coefficients.** In this setting we add control constraints. This is done in a very specific way, namely by adding control constraints for each mode k to the according Fourier coefficients. Again we observe a decoupling with respect to the modes k. Due to the control constraints the optimality system obtains a nonlinear structure. In order to deal with the nonlinearity, following Herzog and Sachs [13], we apply a semi-smooth Newton approach. Consequently, constraints to the Fourier coefficients of the control $\mathbf{u}$ can be incorporated in the same manner as done in [18], in combination with the framework of mixed variational formulations derived in this work. Therefore, we consider the following minimization problem: Minimize the functional

$$J(\mathbf{y}, \mathbf{u}) = \frac{1}{2} \int_{\Omega \times (0,T)} |\mathbf{y} - \mathbf{y_d}|^2 \mathrm{d}\mathbf{x}\, \mathrm{d}t + \frac{\lambda}{2} \int_{\Omega \times (0,T)} |\mathbf{u}|^2 \mathrm{d}\mathbf{x}\, \mathrm{d}t, \tag{24}$$

subject to the state equation

$$\begin{cases} \sigma \dfrac{\partial \mathbf{y}}{\partial t} + \mathbf{curl}(\nu\ \mathbf{curl}\, \mathbf{y}) = \mathbf{u}, & \text{in } \Omega \times (0,T), \\ \mathbf{div}(\sigma \mathbf{y}) = 0, & \text{in } \Omega \times (0,T), \\ \mathbf{y} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega \times (0,T), \\ \mathbf{y}(0) = \mathbf{y}(T), & \text{in } \Omega, \end{cases} \tag{25}$$

with the control constraints associated to the Fourier coefficients

$$\begin{aligned} \underline{\mathbf{u}}_\mathbf{k}^\mathbf{c} \leq \mathbf{u_k^c} \leq \overline{\mathbf{u}}_\mathbf{k}^\mathbf{c}, &\quad \text{a.e. in } \Omega, k = 0, 1, \dots, N, \\ \underline{\mathbf{u}}_\mathbf{k}^\mathbf{s} \leq \mathbf{u_k^s} \leq \overline{\mathbf{u}}_\mathbf{k}^\mathbf{s}, &\quad \text{a.e. in } \Omega, k = 1, \dots, N. \end{aligned}$$

Here $\underline{\mathbf{u}}_\mathbf{k}^\mathbf{c}, \underline{\mathbf{u}}_\mathbf{k}^\mathbf{s}, \overline{\mathbf{u}}_\mathbf{k}^\mathbf{c}, \overline{\mathbf{u}}_\mathbf{k}^\mathbf{s} \in \mathbf{L_2}(\Omega)$ are given data and furthermore $\underline{\mathbf{u}}_\mathbf{k}^\mathbf{c} \leq \overline{\mathbf{u}}_\mathbf{k}^\mathbf{c}$ and $\underline{\mathbf{u}}_\mathbf{k}^\mathbf{s} \leq \overline{\mathbf{u}}_\mathbf{k}^\mathbf{s}$ holds a.e in Ω. Since the problem decouples with respect to the modes k, we again concentrate on one block for a fixed k. For simplicity we drop the subindex k. Therefore we consider the following problem: Minimize the functional

$$J(\mathbf{y^c}, \mathbf{y^s}, \mathbf{u^c}, \mathbf{u^c}) = \frac{1}{2} \sum_{j \in \{c,s\}} \int_\Omega |\mathbf{y^j} - \mathbf{y_d^j}|^2 \mathrm{d}\mathbf{x} + \frac{\lambda}{2} \sum_{j \in \{c,s\}} \int_\Omega |\mathbf{u^j}|^2 \mathrm{d}\mathbf{x},$$

subject to the state equation

$$\begin{cases} k\omega\sigma\, \mathbf{y^s} + \mathbf{curl}(\nu\ \mathbf{curl}\, \mathbf{y^c}) = \mathbf{u^c}, & \text{in } \Omega, \\ -k\omega\, \sigma \mathbf{y^c} + \mathbf{curl}(\nu\ \mathbf{curl}\, \mathbf{y^s}) = \mathbf{u^s}, & \text{in } \Omega, \\ \mathbf{div}(\sigma \mathbf{y^c}) = 0, & \text{in } \Omega, \\ \mathbf{div}(\sigma \mathbf{y^s}) = 0, & \text{in } \Omega, \\ \mathbf{y^c} \times \mathbf{n} = \mathbf{y^s} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega, \end{cases} \tag{26}$$

with the control constraints associated to the Fourier coefficients

$$\begin{aligned} \underline{\mathbf{u}}^{\mathbf{c}} \le \mathbf{u}^{\mathbf{c}} \le \overline{\mathbf{u}}^{\mathbf{c}}, &\quad \text{a.e. in } \Omega, \\ \underline{\mathbf{u}}^{\mathbf{s}} \le \mathbf{u}^{\mathbf{s}} \le \overline{\mathbf{u}}^{\mathbf{s}}, &\quad \text{a.e. in } \Omega. \end{aligned}$$

The first order system of necessary and sufficient optimality conditions can be expressed as follows: Find $(\Upsilon_h, \Psi_h) \in \mathcal{ND}_0(\mathcal{T}_h)^4 \times \mathcal{S}_1(\mathcal{T}_h)^4$, such that

$$\begin{cases} -\omega k(\sigma \mathbf{p}_{\mathbf{h}}^{\mathbf{s}}, \mathbf{v}_{\mathbf{h}}^{\mathbf{c}})_{\mathbf{L_2}(\Omega)} + (\nu \,\mathbf{curl}\, \mathbf{p}_{\mathbf{h}}^{\mathbf{c}}, \mathbf{curl}\, \mathbf{v}_{\mathbf{h}}^{\mathbf{c}})_{\mathbf{L_2}(\Omega)} \\ \qquad + (\mathbf{y}_{\mathbf{h}}^{\mathbf{c}}, \mathbf{v}_{\mathbf{h}}^{\mathbf{c}})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma \mathbf{v}_{\mathbf{h}}^{\mathbf{c}}, \nabla \rho_h^c)_{\mathbf{L_2}(\Omega)} = (\mathbf{y}_{\mathbf{d}}^{\mathbf{c}}, \mathbf{v}_{\mathbf{h}}^{\mathbf{c}})_{\mathbf{L_2}(\Omega)}, \\ \qquad \omega k(\sigma \mathbf{p}_{\mathbf{h}}^{\mathbf{c}}, \nabla \eta_h^c)_{\mathbf{L_2}(\Omega)} = 0, \\ \omega k(\sigma \mathbf{p}_{\mathbf{h}}^{\mathbf{c}}, \mathbf{v}_{\mathbf{h}}^{\mathbf{s}})_{\mathbf{L_2}(\Omega)} + (\nu \,\mathbf{curl}\, \mathbf{p}_{\mathbf{h}}^{\mathbf{s}}, \mathbf{curl}\, \mathbf{v}_{\mathbf{h}}^{\mathbf{s}})_{\mathbf{L_2}(\Omega)} \\ \qquad + (\mathbf{y}_{\mathbf{h}}^{\mathbf{s}}, \mathbf{v}_{\mathbf{h}}^{\mathbf{s}})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma \mathbf{v}_{\mathbf{h}}^{\mathbf{s}}, \nabla \rho_h^s)_{\mathbf{L_2}(\Omega)} = (\mathbf{y}_{\mathbf{d}}^{\mathbf{s}}, \mathbf{v}_{\mathbf{h}}^{\mathbf{s}})_{\mathbf{L_2}(\Omega)}, \\ \qquad \omega k(\sigma \mathbf{p}_{\mathbf{h}}^{\mathbf{s}}, \nabla \eta_h^s)_{\mathbf{L_2}(\Omega)} = 0, \\ \lambda(\mathbf{u}_{\mathbf{h}}^{\mathbf{c}}, \mathbf{w}_{\mathbf{h}}^{\mathbf{c}})_{\mathbf{L_2}(\Omega)} - (\mathbf{p}_{\mathbf{h}}^{\mathbf{c}}, \mathbf{w}_{\mathbf{h}}^{\mathbf{c}})_{\mathbf{L_2}(\Omega)} + (\boldsymbol{\xi}_{\mathbf{h}}^{\mathbf{c}}, \mathbf{w}_{\mathbf{h}}^{\mathbf{c}})_{\mathbf{L_2}(\Omega)} = 0, \\ \lambda(\mathbf{u}_{\mathbf{h}}^{\mathbf{s}}, \mathbf{w}_{\mathbf{h}}^{\mathbf{s}})_{\mathbf{L_2}(\Omega)} - (\mathbf{p}_{\mathbf{h}}^{\mathbf{s}}, \mathbf{w}_{\mathbf{h}}^{\mathbf{s}})_{\mathbf{L_2}(\Omega)} + (\boldsymbol{\xi}_{\mathbf{h}}^{\mathbf{s}}, \mathbf{w}_{\mathbf{h}}^{\mathbf{s}})_{\mathbf{L_2}(\Omega)} = 0, \\ \omega k(\sigma \mathbf{y}_{\mathbf{h}}^{\mathbf{s}}, \mathbf{q}_{\mathbf{h}}^{\mathbf{c}})_{\mathbf{L_2}(\Omega)} + (\nu \,\mathbf{curl}\, \mathbf{y}_{\mathbf{h}}^{\mathbf{c}}, \mathbf{curl}\, \mathbf{q}_{\mathbf{h}}^{\mathbf{c}})_{\mathbf{L_2}(\Omega)} \\ \qquad - (\mathbf{u}_{\mathbf{h}}^{\mathbf{c}}, \mathbf{q}_{\mathbf{h}}^{\mathbf{c}})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma \mathbf{q}_{\mathbf{h}}^{\mathbf{c}}, \nabla \mu_h^c)_{\mathbf{L_2}(\Omega)} = 0, \\ \qquad \omega k(\sigma \mathbf{y}_{\mathbf{h}}^{\mathbf{c}}, \nabla \theta_h^c)_{\mathbf{L_2}(\Omega)} = 0, \\ -\omega k(\sigma \mathbf{y}_{\mathbf{h}}^{\mathbf{c}}, \mathbf{q}_{\mathbf{h}}^{\mathbf{s}})_{\mathbf{L_2}(\Omega)} + (\nu \,\mathbf{curl}\, \mathbf{y}_{\mathbf{h}}^{\mathbf{s}}, \mathbf{curl}\, \mathbf{q}_{\mathbf{h}}^{\mathbf{s}})_{\mathbf{L_2}(\Omega)} \\ \qquad - (\mathbf{u}_{\mathbf{h}}^{\mathbf{s}}, \mathbf{q}_{\mathbf{h}}^{\mathbf{s}})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma \mathbf{q}_{\mathbf{h}}^{\mathbf{s}}, \nabla \mu_h^s)_{\mathbf{L_2}(\Omega)} = 0, \\ \qquad \omega k(\sigma \mathbf{y}_{\mathbf{h}}^{\mathbf{s}}, \nabla \theta_h^s)_{\mathbf{L_2}(\Omega)} = 0, \\ \boldsymbol{\xi}_{\mathbf{h}}^{\mathbf{c}} - \mathbf{max}(\mathbf{0}, \boldsymbol{\xi}_{\mathbf{h}}^{\mathbf{c}} + C(\mathbf{u}_{\mathbf{h}}^{\mathbf{c}} - \overline{\mathbf{u}}^{\mathbf{c}})) - \mathbf{min}(\mathbf{0}, \boldsymbol{\xi}_{\mathbf{h}}^{\mathbf{c}} + C(\underline{\mathbf{u}}^{\mathbf{c}} - \mathbf{u}_{\mathbf{h}}^{\mathbf{c}})) = \mathbf{0}, \\ \boldsymbol{\xi}_{\mathbf{h}}^{\mathbf{s}} - \mathbf{max}(\mathbf{0}, \boldsymbol{\xi}_{\mathbf{h}}^{\mathbf{s}} + C(\mathbf{u}_{\mathbf{h}}^{\mathbf{s}} - \overline{\mathbf{u}}^{\mathbf{s}})) - \mathbf{min}(\mathbf{0}, \boldsymbol{\xi}_{\mathbf{h}}^{\mathbf{s}} + C(\underline{\mathbf{u}}^{\mathbf{s}} - \mathbf{u}_{\mathbf{h}}^{\mathbf{s}})) = \mathbf{0}, \end{cases}$$

for all $(\Phi_h, \Theta_h) \in \mathcal{ND}_0(\mathcal{T}_h)^4 \times \mathcal{S}_1(\mathcal{T}_h)^4$. Here, for any three-dimensional vector function $\mathbf{y} = (y_1, y_2, y_3)$, $\mathbf{max}(0, \mathbf{y})$ is the component wise application of max in the pointwise sense, i.e. $\mathbf{max}(0, \mathbf{y}) := (\max(0, y_1), \max(0, y_2), \max(0, y_3))$. Furthermore C is some positive constant. This system is nonlinear, but due to [14] the last two equations enjoy the Newton differentiability, at least for $C = \lambda$. In order to solve this system, we use the primal-dual active set method as introduced in [14]. This method is equivalent to a semi-smooth Newton method. The strategy proceeds as follows: At each Newton iterate l for $j \in \{c, s\}$ the active sets are determined by

$$\begin{aligned} \mathcal{E}_l^{j,+} &= \{\mathbf{x} \in \Omega : \boldsymbol{\xi}_{\mathbf{h},\mathbf{l}}^{\mathbf{j}} + C(\mathbf{u}_{\mathbf{h},\mathbf{l}}^{\mathbf{j}} - \overline{\mathbf{u}}^{\mathbf{j}}) > 0\}, \\ \mathcal{E}_l^{j,-} &= \{\mathbf{x} \in \Omega : \boldsymbol{\xi}_{\mathbf{h},\mathbf{l}}^{\mathbf{j}} - C(\underline{\mathbf{u}}^{\mathbf{j}} - \mathbf{u}_{\mathbf{h},}^{\mathbf{j}}) < 0\}. \end{aligned}$$

and the inactive sets are $\mathcal{I}_l^j = \Omega \backslash \mathcal{E}_l^j$, where $\mathcal{E}_l^j = \mathcal{E}_l^{j,+} \cup \mathcal{E}_l^{j,-}$. Consequently, the Newton step for the solution of (5.1), given in term of the new iterate, reads as follows (for simplicity the index for the Newton iteration is dropped):

$$\begin{cases} -\omega k(\sigma\mathbf{p_h^s},\mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{p_h^c},\mathbf{curl}\,\mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} \\ \qquad +(\mathbf{y_h^c},\mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma\mathbf{v_h^c},\nabla\rho_h^c)_{\mathbf{L_2}(\Omega)} = (\mathbf{y_d^c},\mathbf{v_h^c})_{\mathbf{L_2}(\Omega)}, \\ \qquad \omega k(\sigma\mathbf{p_h^c},\nabla\eta_h^c)_{\mathbf{L_2}(\Omega)} = 0, \\ \omega k(\sigma\mathbf{p_h^c},\mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{p_h^s},\mathbf{curl}\,\mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} \\ \qquad +(\mathbf{y_h^s},\mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma\mathbf{v_h^s},\nabla\rho_h^s)_{\mathbf{L_2}(\Omega)} = (\mathbf{y_d^s},\mathbf{v_h^s})_{\mathbf{L_2}(\Omega)}, \\ \qquad \omega k(\sigma\mathbf{p_h^s},\nabla\eta_h^s)_{\mathbf{L_2}(\Omega)} = 0, \\ \lambda(\mathbf{u_h^c},\mathbf{w_h^c})_{\mathbf{L_2}(\Omega)} - (\mathbf{p_h^c},\mathbf{w_h^c})_{\mathbf{L_2}(\Omega)} + (\boldsymbol{\xi}_\mathbf{h}^\mathbf{c},\mathbf{w_h^c})_{\mathbf{L_2}(\Omega)} = 0, \\ \lambda(\mathbf{u_h^s},\mathbf{w_h^s})_{\mathbf{L_2}(\Omega)} - (\mathbf{p_h^s},\mathbf{w_h^s})_{\mathbf{L_2}(\Omega)} + (\boldsymbol{\xi}_\mathbf{h}^\mathbf{s},\mathbf{w_h^s})_{\mathbf{L_2}(\Omega)} = 0, \\ \omega k(\sigma\mathbf{y_h^s},\mathbf{q_h^c})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{y_h^c},\mathbf{curl}\,\mathbf{q_h^c})_{\mathbf{L_2}(\Omega)} \\ \qquad -(\mathbf{u_h^c},\mathbf{q_h^c})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma\mathbf{q_h^c},\nabla\mu_h^c)_{\mathbf{L_2}(\Omega)} = 0, \\ \qquad \omega k(\sigma\mathbf{y_h^c},\nabla\theta_h^c)_{\mathbf{L_2}(\Omega)} = 0, \\ -\omega k(\sigma\mathbf{y_h^c},\mathbf{q_h^s})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{y_h^s},\mathbf{curl}\,\mathbf{q_h^s})_{\mathbf{L_2}(\Omega)} \\ \qquad -(\mathbf{u_h^s},\mathbf{q_h^s})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma\mathbf{q_h^s},\nabla\mu_h^s)_{\mathbf{L_2}(\Omega)} = 0, \\ \qquad \omega k(\sigma\mathbf{y_h^s},\nabla\theta_h^s)_{\mathbf{L_2}(\Omega)} = 0, \\ C\chi_{\mathcal{E}^c}\mathbf{u_h^c} + \chi_{\mathcal{I}^c}\boldsymbol{\xi}_\mathbf{h}^\mathbf{c} - C(\chi_{\mathcal{E}^{c},+}\overline{\mathbf{u}}^\mathbf{c} + \chi_{\mathcal{E}^{c},-}\underline{\mathbf{u}}^\mathbf{c}) = \mathbf{0}, \\ C\chi_{\mathcal{E}^s}\mathbf{u_h^s} + \chi_{\mathcal{I}^s}\boldsymbol{\xi}_\mathbf{h}^\mathbf{s} - C(\chi_{\mathcal{E}^{s},+}\overline{\mathbf{u}}^\mathbf{s} + \chi_{\mathcal{E}^{s},-}\underline{\mathbf{u}}^\mathbf{s}) = \mathbf{0}. \end{cases}$$

The symbol χ denotes the characteristic function with respect to the set denoted in the subscript. Next we use, that the restriction of $\boldsymbol{\xi}_\mathbf{h}^\mathbf{j}$, $j \in \{c, s\}$, to the inactive sets $\mathcal{I}^j$ is zero, and hence this variable can be eliminated from the system. Therefore, we introduce the new variables $\boldsymbol{\xi}_{\mathbf{h},\boldsymbol{\mathcal{E}}}^\mathbf{j}$, namely the restriction of $\boldsymbol{\xi}_\mathbf{h}^\mathbf{j}$ to the active set $\mathcal{E}^j$. Furthermore, we derive the weak formulation of the last two equations, by multiplying with test functions $\mathbf{z}_{\mathbf{h},\boldsymbol{\mathcal{E}}}^\mathbf{j} \in \mathbf{L_2}(\mathcal{E}^j)$. Consequently, we are dealing with the following problem:

$$\begin{cases} -\omega k(\sigma\mathbf{p_h^s},\mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{p_h^c},\mathbf{curl}\,\mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} \\ \qquad +(\mathbf{y_h^c},\mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma\mathbf{v_h^c},\nabla\rho_h^c)_{\mathbf{L_2}(\Omega)} = (\mathbf{y_d^c},\mathbf{v_h^c})_{\mathbf{L_2}(\Omega)}, \\ \qquad \omega k(\sigma\mathbf{p_h^c},\nabla\eta_h^c)_{\mathbf{L_2}(\Omega)} = 0, \\ \omega k(\sigma\mathbf{p_h^c},\mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{p_h^s},\mathbf{curl}\,\mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} \\ \qquad +(\mathbf{y_h^s},\mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma\mathbf{v_h^s},\nabla\rho_h^s)_{\mathbf{L_2}(\Omega)} = (\mathbf{y_d^s},\mathbf{v_h^s})_{\mathbf{L_2}(\Omega)}, \\ \qquad \omega k(\sigma\mathbf{p_h^s},\nabla\eta_h^s)_{\mathbf{L_2}(\Omega)} = 0, \\ \lambda(\mathbf{u_h^c},\mathbf{w_h^c})_{\mathbf{L_2}(\Omega)} - (\mathbf{p_h^c},\mathbf{w_h^c})_{\mathbf{L_2}(\Omega)} + (\boldsymbol{\xi}_{\mathbf{h},\boldsymbol{\mathcal{E}}}^\mathbf{c},\mathcal{P}_{\mathcal{E}^c}\mathbf{w_h^c})_{\mathbf{L_2}(\mathcal{E}_h^c)} = 0, \\ \lambda(\mathbf{u_h^s},\mathbf{w_h^s})_{\mathbf{L_2}(\Omega)} - (\mathbf{p_h^s},\mathbf{w_h^s})_{\mathbf{L_2}(\Omega)} + (\boldsymbol{\xi}_{\mathbf{h},\boldsymbol{\mathcal{E}}}^\mathbf{s},\mathcal{P}_{\mathcal{E}^s}\mathbf{w_h^s})_{\mathbf{L_2}(\mathcal{E}_h^c)} = 0, \\ \omega k(\sigma\mathbf{y_h^s},\mathbf{q_h^c})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{y_h^c},\mathbf{curl}\,\mathbf{q_h^c})_{\mathbf{L_2}(\Omega)} \\ \qquad -(\mathbf{u_h^c},\mathbf{q_h^c})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma\mathbf{q_h^c},\nabla\mu_h^c)_{\mathbf{L_2}(\Omega)} = 0, \\ \qquad \omega k(\sigma\mathbf{y_h^c},\nabla\theta_h^c)_{\mathbf{L_2}(\Omega)} = 0, \\ -\omega k(\sigma\mathbf{y_h^c},\mathbf{q_h^s})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{y_h^s},\mathbf{curl}\,\mathbf{q_h^s})_{\mathbf{L_2}(\Omega)} \\ \qquad -(\mathbf{u_h^s},\mathbf{q_h^s})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma\mathbf{q_h^s},\nabla\mu_h^s)_{\mathbf{L_2}(\Omega)} = 0, \\ \qquad \omega k(\sigma\mathbf{y_h^s},\nabla\theta_h^s)_{\mathbf{L_2}(\Omega)} = 0, \\ (\mathcal{P}_{\mathcal{E}^c}\mathbf{u_h^c},\mathbf{z}_{\mathbf{h},\boldsymbol{\mathcal{E}}}^\mathbf{c})_{\mathbf{L_2}(\mathcal{E}_h^c)} - (\mathcal{P}_{\mathcal{E}^{c},+}\overline{\mathbf{u}}^\mathbf{c} + \mathcal{P}_{\mathcal{E}^{c},-}\underline{\mathbf{u}}^\mathbf{c},\mathbf{z}_{\mathbf{h},\boldsymbol{\mathcal{E}}}^\mathbf{c})_{\mathbf{L_2}(\mathcal{E}^c)} = 0, \\ (\mathcal{P}_{\mathcal{E}^s}\mathbf{u_h^s},\mathbf{z}_{\mathbf{k},\boldsymbol{\mathcal{E}}}^\mathbf{s})_{\mathbf{L_2}(\mathcal{E}^s)} - (\mathcal{P}_{\mathcal{E}^{s},+}\overline{\mathbf{u}}^\mathbf{s} + \mathcal{P}_{\mathcal{E}^{s},-}\underline{\mathbf{u}}^\mathbf{s},\mathbf{z}_{\mathbf{h},\boldsymbol{\mathcal{E}}}^\mathbf{s})_{\mathbf{L_2}(\mathcal{E}^s)} = 0. \end{cases}$$

Here the projections $\mathcal{P}_{\mathcal{X}}$, $\mathcal{X} \in \{\mathcal{E}^{c,+}, \mathcal{E}^{c,-}, \mathcal{E}^{c}, \mathcal{E}^{s,+}, \mathcal{E}^{s,-}, \mathcal{E}^{s}\}$, are defined by

$$\begin{aligned}
&\mathcal{P}_{\mathcal{X}} : \mathbf{L_2}(\Omega) \to \mathbf{L_2}(\mathcal{X}),\\
&(\mathcal{P}_{\mathcal{X}}\mathbf{u}, \mathbf{v})_{\mathbf{L_2}(\mathcal{X})} = (\mathbf{u}, \mathbf{v})_{\mathbf{L_2}(\mathcal{X})}, \quad \forall \mathbf{u} \in \mathbf{L_2}(\Omega), \mathbf{v} \in \mathbf{L_2}(\mathcal{X}).
\end{aligned}$$

The adjoint operators $\mathcal{P}^*_{\mathcal{X}}$ are defined by

$$\begin{aligned}
&\mathcal{P}^*_{\mathcal{X}} : \mathbf{L_2}(\mathcal{X}) \to \mathbf{L_2}(\Omega),\\
&(\mathbf{u}, \mathcal{P}^*_{\mathcal{X}}\mathbf{v})_{\mathbf{L_2}(\Omega)} = (\mathcal{P}_{\mathcal{X}}\mathbf{u}, \mathbf{v})_{\mathbf{L_2}(\mathcal{X})}, \quad \forall \mathbf{u} \in \mathbf{L_2}(\Omega), \mathbf{v} \in \mathbf{L_2}(\mathcal{X}).
\end{aligned}$$

In the usual manner, the control variables $\mathbf{u^c_h}$ and $\mathbf{u^s_h}$ can be eliminated, using the fifth and the sixth equation. Finally, we eliminate $\boldsymbol{\xi}^{\mathbf{c}}_{\mathbf{h},\boldsymbol{\mathcal{E}}}$ and $\boldsymbol{\xi}^{\mathbf{s}}_{\mathbf{h},\boldsymbol{\mathcal{E}}}$ and end up with the reduced optimality system: Find $(\Upsilon_h, \Psi_h) \in \mathcal{ND}_0(\mathcal{T}_h)^4 \times \mathcal{S}_1(\mathcal{T}_h)^4$, such that

$$\left\{\begin{aligned}
&-\omega k(\sigma \mathbf{p^s_h}, \mathbf{v^c_h})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{p^c_h}, \mathbf{curl}\,\mathbf{v^c_h})_{\mathbf{L_2}(\Omega)}\\
&\qquad\qquad + (\mathbf{y^c_h}, \mathbf{v^c_h})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma \mathbf{v^c_h}, \nabla \rho^c_h)_{\mathbf{L_2}(\Omega)} = (\mathbf{y^c_d}, \mathbf{v^c_h})_{\mathbf{L_2}(\Omega)},\\
&\qquad\qquad \omega k(\sigma \mathbf{p^c_h}, \nabla \eta^c_h)_{\mathbf{L_2}(\Omega)} = 0,\\
&\omega k(\sigma \mathbf{p^c_h}, \mathbf{v^s_h})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{p^s_h}, \mathbf{curl}\,\mathbf{v^s_h})_{\mathbf{L_2}(\Omega)}\\
&\qquad\qquad + (\mathbf{y^s_h}, \mathbf{v^s_h})_{\mathbf{L_2}(\Omega)} + \omega k(\sigma \mathbf{v^s_h}, \nabla \rho^s_h)_{\mathbf{L_2}(\Omega)} = (\mathbf{y^s_d}, \mathbf{v^s_h})_{\mathbf{L_2}(\Omega)},\\
&\qquad\qquad \omega k(\sigma \mathbf{p^s_h}, \nabla \eta^s_h)_{\mathbf{L_2}(\Omega)} = 0,\\
&\omega k(\sigma \mathbf{y^s_h}, \mathbf{q^c_h})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{y^c_h}, \mathbf{curl}\,\mathbf{q^c_h})_{\mathbf{L_2}(\Omega)}\\
&-\frac{1}{\lambda}\left[(\mathbf{p^c_h}, \mathbf{q^c_h})_{\mathbf{L_2}(\Omega)} - (\mathcal{P}^*_{\mathcal{E}^c}\mathcal{P}_{\mathcal{E}^c}\mathbf{p^c_h}, \mathbf{q^c_h})_{\mathbf{L_2}(\Omega)}\right] + \omega k(\sigma \mathbf{q^c_h}, \nabla \mu^c_h)_{\mathbf{L_2}(\Omega)} = (\mathbf{f^c_h}, \mathbf{q^c_h})_{\mathbf{L_2}(\Omega)},\\
&\qquad\qquad \omega k(\sigma \mathbf{y^c_h}, \nabla \theta^c_h)_{\mathbf{L_2}(\Omega)} = 0,\\
&-\omega k(\sigma \mathbf{y^c_h}, \mathbf{q^s_h})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{y^s_h}, \mathbf{curl}\,\mathbf{q^s_h})_{\mathbf{L_2}(\Omega)}\\
&-\frac{1}{\lambda}\left[(\mathbf{p^s_h}, \mathbf{q^s_h})_{\mathbf{L_2}(\Omega)} - (\mathcal{P}^*_{\mathcal{E}^s}\mathcal{P}_{\mathcal{E}^s}\mathbf{p^s_h}, \mathbf{q^s_h})_{\mathbf{L_2}(\Omega)}\right] + \omega k(\sigma \mathbf{q^s_h}, \nabla \mu^s_h)_{\mathbf{L_2}(\Omega)} = (\mathbf{f^s_h}, \mathbf{q^s_h})_{\mathbf{L_2}(\Omega)},\\
&\qquad\qquad \omega k(\sigma \mathbf{y^s_h}, \nabla \theta^s_h)_{\mathbf{L_2}(\Omega)} = 0.
\end{aligned}\right.$$

for all $(\Phi_h, \Theta_h) \in \mathcal{ND}_0(\mathcal{T}_h)^4 \times \mathcal{S}_1(\mathcal{T}_h)^4$. Here $\mathbf{f^j_h}$, for $j \in \{c, s\}$, is given by

$$\mathbf{f^j_h} = \mathcal{P}^*_{\mathcal{E}^j}(\mathcal{P}_{\mathcal{E}^{j,+}}\overline{\mathbf{u}}^{\mathbf{j}} + \mathcal{P}_{\mathcal{E}^{j,-}}\underline{\mathbf{u}}^{\mathbf{j}}).$$

In this setting we have

$$\mathcal{A}_3((\Upsilon, \Psi), (\Phi, \Theta)) := a^3(\Upsilon, \Phi) + b(\Phi, \Psi) + b(\Upsilon, \Theta)$$

with

$$\begin{aligned}
a^3((\mathbf{y^c}, \mathbf{y^s}, \mathbf{p^c}, \mathbf{p^s}), (\mathbf{v^c}, \mathbf{v^s}, \mathbf{q^c}, \mathbf{q^s})) &:= a_a((\mathbf{y^c}, \mathbf{y^s}), (\mathbf{v^c}, \mathbf{v^s}))\\
&+ a_b((\mathbf{v^c}, \mathbf{v^s}), (\mathbf{p^c}, \mathbf{p^s})) + a_b((\mathbf{y^c}, \mathbf{y^s}), (\mathbf{q^c}, \mathbf{q^s})) - a^3_c((\mathbf{p^c}, \mathbf{p^s}), (\mathbf{q^c}, \mathbf{q^s}))
\end{aligned}$$

and the new bilinear form

$$a^3_c((\mathbf{y^c}, \mathbf{y^s}), (\mathbf{v^c}, \mathbf{v^s})) := \sum_{j \in \{c,s\}} (\mathbf{p^j}, \mathbf{q^j})_{\mathbf{L_2}(\Omega)} - (\mathcal{P}^*_{\mathcal{E}^j}\mathcal{P}_{\mathcal{E}^j}\mathbf{p^j}, \mathbf{q^j})_{\mathbf{L_2}(\Omega)}.$$

The main result is summarized in the following Lemma, that claims that the inf-sup condition and the sup-sup condition in the Theorem of Babuška-Aziz are fulfilled in the non-standard norm.

Lemma 6. *We have*

$$\underline{c}_3 \|(\Upsilon_h, \Psi_h)\|_{\mathcal{P}} \le \sup_{(\Phi_h, \Theta_h) \neq \mathbf{0}} \frac{\mathcal{A}_3((\Upsilon_h, \Psi_h), (\Phi_h, \Theta_h))}{\|(\Phi_h, \Theta_h)\|_{\mathcal{P}}} \le \overline{c}_3 \|(\Upsilon_h, \Psi_h)\|_{\mathcal{P}}$$

for all $(\Upsilon_h, \Psi_h) \in \mathcal{ND}_0(\mathcal{T}_h)^4 \times \mathcal{S}_1(\mathcal{T}_h)^4$*. Here the constants* $\underline{c}_3$*,* $\overline{c}_3$ *are independent of* h*,* k*,* ω *and the active sets* $\mathcal{E}^c$ *and* $\mathcal{E}^s$*.*

Proof. The proof is basically the same as the proof of Lemma 4. The main differences are the lower and upper bounds for $a_c^3(\cdot,\cdot)$. We have

$$0 \leq a_c^3((\mathbf{p_h^c}, \mathbf{p_h^s}), (\mathbf{p_h^c}, \mathbf{p_h^s})) \leq 2\|(\mathbf{p_h^c}, \mathbf{p_h^s})\|_{\mathcal{C}_2}^2.$$

This completes the proof. □

5.2. **State constraints to the Fourier coefficients.** In this setting we add state constraints. This is done in a very specific way, namely by adding state constraints for each mode k to the according Fourier coefficients. Therefore, we consider the following minimization problem: Minimize the functional

$$J(\mathbf{y}, \mathbf{u}) = \frac{1}{2} \int_{\Omega\times(0,T)} |\mathbf{y} - \mathbf{y_d}|^2 \mathrm{d}\mathbf{x}\, \mathrm{dt} + \frac{\lambda}{2} \int_{\Omega\times(0,T)} |\mathbf{u}|^2 \mathrm{d}\mathbf{x}\, \mathrm{dt}, \tag{27}$$

subject to the state equation

$$\begin{cases} \sigma \dfrac{\partial \mathbf{y}}{\partial t} + \mathbf{curl}(\nu\ \mathbf{curl}\, \mathbf{y}) = \mathbf{u}, & \text{in } \Omega \times (0,T), \\ \mathbf{y} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega \times (0,T), \\ \mathbf{y}(0) = \mathbf{y}(T), & \text{in } \Omega, \end{cases} \tag{28}$$

with the state constraints associated to the Fourier coefficients

$$\begin{aligned} \underline{\mathbf{y}}_\mathbf{k}^\mathbf{c} \leq \mathbf{y_k^c} \leq \overline{\mathbf{y}}_\mathbf{k}^\mathbf{c}, &\quad \text{a.e. in } \Omega, k = 0, 1, \dots, N, \\ \underline{\mathbf{y}}_\mathbf{k}^\mathbf{s} \leq \mathbf{y_k^s} \leq \overline{\mathbf{y}}_\mathbf{k}^\mathbf{s}, &\quad \text{a.e. in } \Omega, k = 1, \dots, N. \end{aligned}$$

Here $\underline{\mathbf{y}}_\mathbf{k}^\mathbf{c}, \underline{\mathbf{y}}_\mathbf{k}^\mathbf{s}, \overline{\mathbf{y}}_\mathbf{k}^\mathbf{c}, \overline{\mathbf{y}}_\mathbf{k}^\mathbf{s} \in \mathbf{L_2}(\Omega)$ are given data and furthermore $\underline{\mathbf{y}}_\mathbf{k}^\mathbf{c} \leq \overline{\mathbf{y}}_\mathbf{k}^\mathbf{c}$ and $\underline{\mathbf{y}}_\mathbf{k}^\mathbf{s} \leq \overline{\mathbf{y}}_\mathbf{k}^\mathbf{s}$ holds a.e in Ω. Again we observe a decoupling with respect to the modes k, and therefore drop the mode subindex k. In order to incorporate the state constraints, we follow the analytical framework presented in [34]. Due to the lack of regularity of the control to state map, a penalization method, that is also called Moreau-Yosida regularization, has to be used. Therefore the regularized problem reads as follows: Minimize the functional

$$\begin{aligned} J^\varepsilon(\mathbf{y^c}, \mathbf{y^s}, \mathbf{u^c}, \mathbf{u^c}) = {} & \frac{1}{2} \sum_{j\in\{c,s\}} \int_\Omega |\mathbf{y^j} - \mathbf{y_d^j}|^2 \mathrm{d}\mathbf{x} + \frac{\lambda}{2} \sum_{j\in\{c,s\}} \int_\Omega |\mathbf{u^j}|^2 \mathrm{d}\mathbf{x} \\ & + \frac{1}{2\varepsilon} \sum_{j\in\{c,s\}} \left(\|\mathbf{max}(0, \mathbf{y^j} - \overline{\mathbf{y}}^\mathbf{j})\|_{\mathbf{L_2}(\Omega)}^2 + \|\mathbf{min}(0, \mathbf{y^j} - \underline{\mathbf{y}}^\mathbf{j})\|_{\mathbf{L_2}(\Omega)}^2 \right), \end{aligned} \tag{29}$$

subject to the state equation

$$\begin{cases} k\omega\sigma\, \mathbf{y^s} + \mathbf{curl}(\nu\ \mathbf{curl}\, \mathbf{y^c}) = \mathbf{u^c}, & \text{in } \Omega, \\ -k\omega\, \sigma \mathbf{y^c} + \mathbf{curl}(\nu\ \mathbf{curl}\, \mathbf{y^s}) = \mathbf{u^s}, & \text{in } \Omega, \\ \mathbf{y^c} \times \mathbf{n} = \mathbf{y^s} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega. \end{cases} \tag{30}$$

The first order system of necessary and sufficient optimality conditions of (29)-(30) can be expressed as follows

$$\begin{cases} -\omega k(\sigma \mathbf{p_h^s}, \mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} + (\nu \,\mathbf{curl}\, \mathbf{p_h^c}, \mathbf{curl}\, \mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} \\ \qquad + (\mathbf{y_h^c}, \mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} + (\boldsymbol{\xi}_\mathbf{h}^\mathbf{c}, \mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} = (\mathbf{y_d^c}, \mathbf{v_h^c})_{\mathbf{L_2}(\Omega)}, \\ \omega k(\sigma \mathbf{p_h^c}, \mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} + (\nu \,\mathbf{curl}\, \mathbf{p_h^s}, \mathbf{curl}\, \mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} \\ \qquad + (\mathbf{y_h^s}, \mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} + (\boldsymbol{\xi}_\mathbf{h}^\mathbf{s}, \mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} = (\mathbf{y_d^s}, \mathbf{v_h^s})_{\mathbf{L_2}(\Omega)}, \\ \omega k(\sigma \mathbf{y_h^s}, \mathbf{q_h^c})_{\mathbf{L_2}(\Omega)} + (\nu \,\mathbf{curl}\, \mathbf{y_h^c}, \mathbf{curl}\, \mathbf{q_h^c})_{\mathbf{L_2}(\Omega)} - \lambda^{-1}(\mathbf{p_h^c}, \mathbf{q_h^c})_{\mathbf{L_2}(\Omega)} = 0, \\ -\omega k(\sigma \mathbf{y_h^c}, \mathbf{q_h^s})_{\mathbf{L_2}(\Omega)} + (\nu \,\mathbf{curl}\, \mathbf{y_h^s}, \mathbf{curl}\, \mathbf{q_h^s})_{\mathbf{L_2}(\Omega)} - \lambda^{-1}(\mathbf{p_h^s}, \mathbf{q_h^s})_{\mathbf{L_2}(\Omega)} = 0, \\ \boldsymbol{\xi}_\mathbf{h}^\mathbf{c} - \frac{1}{\varepsilon}\mathbf{max}(\mathbf{0}, \mathbf{y_h^c} - \overline{\mathbf{y}}^\mathbf{c}) + \frac{1}{\varepsilon}\mathbf{min}(\mathbf{0}, \underline{\mathbf{y}}^\mathbf{c} - \mathbf{y_h^c}) = \mathbf{0}, \\ \boldsymbol{\xi}_\mathbf{h}^\mathbf{s} - \frac{1}{\varepsilon}\mathbf{max}(\mathbf{0}, \mathbf{y_h^s} - \overline{\mathbf{y}}^\mathbf{s}) + \frac{1}{\varepsilon}\mathbf{min}(\mathbf{0}, \underline{\mathbf{y}}^\mathbf{s} - \mathbf{y_h^s}) = \mathbf{0}. \end{cases}$$

Again, we use a primal-dual active set strategy. Therefore, at each Newton step l (for simplicity the index for the Newton iteration is skipped), we have to solve the variational problem: Find $(\Upsilon_h, \Psi_h) \in \mathcal{ND}_0(\mathcal{T}_h)^4 \times \mathcal{S}_1(\mathcal{T}_h)^4$, such that

$$\begin{cases} -\omega k(\sigma \mathbf{p_h^s}, \mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} + (\nu \,\mathbf{curl}\, \mathbf{p_h^c}, \mathbf{curl}\, \mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} + (\mathbf{y_h^c}, \mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} + \frac{1}{\varepsilon}(\mathbf{y_h^c}, \mathbf{v_h^c})_{\mathbf{L_2}(\mathcal{E}^c)} \\ \qquad = (\mathbf{y_d^c}, \mathbf{v_h^c})_{\mathbf{L_2}(\Omega)} + \frac{1}{\varepsilon}\left(\chi_{\mathcal{E}^{c,+}}\overline{\mathbf{y}}^\mathbf{c} + \chi_{\mathcal{E}^{c,-}}\underline{\mathbf{y}}^\mathbf{c}\right), \\ \omega k(\sigma \mathbf{p_h^c}, \mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} + (\nu \,\mathbf{curl}\, \mathbf{p_h^s}, \mathbf{curl}\, \mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} + (\mathbf{y_h^s}, \mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} + \frac{1}{\varepsilon}(\mathbf{y_h^s}, \mathbf{v_h^s})_{\mathbf{L_2}(\mathcal{E}^s)} \\ \qquad = (\mathbf{y_d^s}, \mathbf{v_h^s})_{\mathbf{L_2}(\Omega)} + \frac{1}{\varepsilon}\left(\chi_{\mathcal{E}^{s,+}}\overline{\mathbf{y}}^\mathbf{s} + \chi_{\mathcal{E}^{s,-}}\underline{\mathbf{y}}^\mathbf{s}\right), \\ \omega k(\sigma \mathbf{y_h^s}, \mathbf{q_h^c})_{\mathbf{L_2}(\Omega)} + (\nu \,\mathbf{curl}\, \mathbf{y_h^c}, \mathbf{curl}\, \mathbf{q_h^c})_{\mathbf{L_2}(\Omega)} - \lambda^{-1}(\mathbf{p_h^c}, \mathbf{q_h^c})_{\mathbf{L_2}(\Omega)} = 0, \\ -\omega k(\sigma \mathbf{y_h^c}, \mathbf{q_h^s})_{\mathbf{L_2}(\Omega)} + (\nu \,\mathbf{curl}\, \mathbf{y_h^s}, \mathbf{curl}\, \mathbf{q_h^s})_{\mathbf{L_2}(\Omega)} - \lambda^{-1}(\mathbf{p_h^s}, \mathbf{q_h^s})_{\mathbf{L_2}(\Omega)} = 0, \end{cases}$$

for all $(\Phi_h, \Theta_h) \in \mathcal{ND}_0(\mathcal{T}_h)^4 \times \mathcal{S}_1(\mathcal{T}_h)^4$. Here the active sets for the cosine and the sine components, i.e. $j \in \{c, s\}$, are given by

$$\mathcal{E}^{j,+} = \{\mathbf{x} \in \Omega : \mathbf{y_h^j} - \overline{\mathbf{y}}^\mathbf{j} > 0\} \quad \text{and} \quad \mathcal{E}^{j,-} = \{\mathbf{x} \in \Omega : \underline{\mathbf{y}}^\mathbf{j} - \mathbf{y_h^j} < 0\}.$$

The full active sets for $j \in \{c, s\}$ are denoted by $\mathcal{E}^j = \mathcal{E}^{j,+} \cup \mathcal{E}^{j,-}$. According to the notation in Section 2, we introduce the bilinear form

$$\begin{aligned} \mathcal{A}_4(\Upsilon, \Phi) : = {} & a_a^4((\mathbf{y^c}, \mathbf{y^s}), (\mathbf{v^c}, \mathbf{v^s})) + a_b((\mathbf{v^c}, \mathbf{v^s}), (\mathbf{p^c}, \mathbf{p^s})) \\ & + a_b((\mathbf{y^c}, \mathbf{y^s}), (\mathbf{q^c}, \mathbf{q^s})) - a_c((\mathbf{p^c}, \mathbf{p^s}), (\mathbf{q^c}, \mathbf{q^s})) \end{aligned}$$

with $a_b(\cdot,\cdot)$ and $a_c(\cdot,\cdot)$ defined as in (10) and $a_a^4(\cdot,\cdot)$ given by

$$a_a^4((\mathbf{y^c}, \mathbf{y^s}), (\mathbf{v^c}, \mathbf{v^s})) := \sum_{j \in \{c,s\}} (\mathbf{y^j}, \mathbf{v^j})_{\mathbf{L_2}(\Omega)} + \frac{1}{\varepsilon}(\mathbf{y^j}, \mathbf{v^j})_{\mathbf{L_2}(\mathcal{E}^j)}.$$

The main result is summarized in the following Lemma, that claims that the inf-sup condition and the sup-sup condition in the Theorem of Babuška-Aziz are fulfilled in the non-standard norm.

Lemma 7. *We have*

$$\underline{c}_4 \|\Upsilon_h\|_{\mathcal{C}} \le \sup_{\Phi_h \neq 0} \frac{\mathcal{A}_4(\Upsilon_h, \Phi_h)}{\|\Phi_h\|_{\mathcal{C}}} \le \overline{c}_4 \|\Upsilon_h\|_{\mathcal{C}}$$

for all $\Upsilon_h \in \mathcal{ND}_0(\mathcal{T}_h)^4$. Here the constants $\underline{c}_4$, $\overline{c}_4$ are independent of h, k, ω, σ, ν, λ and the active sets $\mathcal{E}^c$ and $\mathcal{E}^s$.

Proof. From [23, Theorem 2], we obtain that the bilinear form $a(\cdot,\cdot)$ is bounded with constant 1 and satisfies an inf-sup condition on the kernel of $b(\cdot,\cdot)$ with constant $1/\sqrt{3}$. Therefore for $a(\cdot,\cdot)$, the two conditions (13) and (14) of Theorem 1 are fulfilled. Now the proof for $\mathcal{A}_4(\cdot,\cdot)$ immediately follows, since

$$a_a^4((\mathbf{y^c},\mathbf{y^s}),(\mathbf{v^c},\mathbf{v^s})) = a_a((\mathbf{y^c},\mathbf{y^s}),(\mathbf{v^c},\mathbf{v^s})) + \frac{1}{\varepsilon}\sum_{j\in\{c,s\}}(\mathbf{y^j},\mathbf{v^j})_{\mathbf{L_2}(\mathcal{E}^j)}$$

and

$$a_a((\mathbf{y^c},\mathbf{y^s}),(\mathbf{y^c},\mathbf{y^s})) \le a_a^4((\mathbf{y^c},\mathbf{y^s}),(\mathbf{y^c},\mathbf{y^s})) \le (1+\frac{1}{\varepsilon})a_a((\mathbf{y^c},\mathbf{y^s}),(\mathbf{y^c},\mathbf{y^s})).$$

This finishes the proof. □

6. Constraints on the final time

In this section, we want to additionally control the end time of the state $\mathbf{y}$. Consequently, we obtain the following minimization problem: Minimize the functional

(31)
$$J(\mathbf{y},\mathbf{u}) = \frac{1}{2}\int_{\Omega\times(0,T)}|\mathbf{y}-\mathbf{y_d}|^2\mathrm{d}\mathbf{x}\,\mathrm{d}t + \frac{\lambda}{2}\int_{\Omega\times(0,T)}|\mathbf{u}|^2\mathrm{d}\mathbf{x}\,\mathrm{d}t + \frac{\alpha}{2}\int_{\Omega}|\mathbf{y}(T)-\mathbf{y_T}|^2\mathrm{d}\mathbf{x},$$

subject to the state equation

$$(32)\qquad \begin{cases} \sigma\dfrac{\partial\mathbf{y}}{\partial t} + \mathbf{curl}(\nu\ \mathbf{curl}\,\mathbf{y}) = \mathbf{u}, & \text{in } \Omega\times(0,T),\\ \mathbf{y}\times\mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega\times(0,T),\\ \mathbf{y}(0) = \mathbf{y}(T), & \text{in } \Omega. \end{cases}$$

Using the multiharmonic representation of the state $\mathbf{y}$, the desired state $\mathbf{y_d}$ and the control $\mathbf{u}$, we can state the minimization problem in the frequency domain: Minimize the functional

(33)
$$J_N(\mathbf{y},\mathbf{u}) = \sum_{j\in\{c,s\}}\sum_{k=0}^{N}\frac{1}{2}\int_\Omega|\mathbf{y_k^j}-\mathbf{y_{d,k}^j}|^2\mathrm{d}\mathbf{x} + \frac{\lambda}{2}\int_\Omega|\mathbf{u_k^j}|^2\mathrm{d}\mathbf{x} + \frac{\alpha}{2}\int_\Omega|\sum_{k=0}^{N}\mathbf{y_k^c}-\mathbf{y_T}|^2\mathrm{d}\mathbf{x},$$

subject to the state equation

$$(34)\quad \text{For } k=0,\dots,N:\ \begin{cases} k\omega\sigma\,\mathbf{y_k^s} + \mathbf{curl}(\nu\ \mathbf{curl}\,\mathbf{y_k^c}) = \mathbf{u_k^c}, & \text{in } \Omega\times(0,T),\\ -k\omega\,\sigma\mathbf{y_k^c} + \mathbf{curl}(\nu\ \mathbf{curl}\,\mathbf{y_k^s}) = \mathbf{u_k^s}, & \text{in } \Omega\times(0,T),\\ \mathbf{y_k^c}\times\mathbf{n} = \mathbf{y_k^s}\times\mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega\times(0,T), \end{cases}$$

The optimality system of (33)-(34) is given by: For each mode $k=0,1,\dots,N$, find the Fourier coefficients $(\mathbf{y_k^c},\mathbf{y_k^s},\mathbf{p_k^c},\mathbf{p_k^s})\in\mathbf{H_0}(\mathbf{curl},\Omega)^4$, such that

$$\begin{cases} (1+\alpha)(\mathbf{y_k^c},\mathbf{v_k^c})_{\mathbf{L_2}(\Omega)} - \omega(\sigma\mathbf{p_k^s},\mathbf{v_k^c})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{p_k^c},\mathbf{curl}\,\mathbf{v_k^c})_{\mathbf{L_2}(\Omega)} \\ \qquad +\alpha\sum_{l\neq k}(\mathbf{y_l^c},\mathbf{v_k^c})_{\mathbf{L_2}(\Omega)} = (\mathbf{y_{d,k}^c},\mathbf{v_k^c})_{\mathbf{L_2}(\Omega)} + \alpha(\mathbf{y_T},\mathbf{v_k^c})_{\mathbf{L_2}(\Omega)},\\ (\mathbf{y_k^s},\mathbf{v_k^s})_{\mathbf{L_2}(\Omega)} + \omega(\sigma\mathbf{p_k^c},\mathbf{v_k^s})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{p_k^s},\mathbf{curl}\,\mathbf{v_k^s})_{\mathbf{L_2}(\Omega)} = (\mathbf{y_{d,k}^s},\mathbf{v_k^s})_{\mathbf{L_2}(\Omega)},\\ \omega(\sigma\mathbf{y_k^s},\mathbf{q_k^c})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{y_k^c},\mathbf{curl}\,\mathbf{q_k^c})_{\mathbf{L_2}(\Omega)} - \dfrac{1}{\lambda}(\mathbf{p_k^c},\mathbf{q_k^c})_{\mathbf{L_2}(\Omega)} = 0,\\ -\omega(\sigma\mathbf{y_k^c},\mathbf{q_k^s})_{\mathbf{L_2}(\Omega)} + (\nu\,\mathbf{curl}\,\mathbf{y_k^s},\mathbf{curl}\,\mathbf{q_k^s})_{\mathbf{L_2}(\Omega)} - \dfrac{1}{\lambda}(\mathbf{p_k^s},\mathbf{q_k^s})_{\mathbf{L_2}(\Omega)} = 0, \end{cases}$$

for all test functions $(\mathbf{v_k^c},\mathbf{v_k^s},\mathbf{q_k^c},\mathbf{q_k^s})\in\mathbf{H_0}(\mathbf{curl},\Omega)^4$. Due to the control of the final time, there is a coupling through the cosine terms $\mathbf{u_k^c}$ of the state. Nevertheless, the resulting system of equations fits into our framework. After the usual finite element discretization with edge elements of lowest order, we end up with the variational formulation: Find $\mathbf{\Upsilon_h} = (\Upsilon_0, \dots, \Upsilon_N) \in \mathcal{ND}_0(\mathcal{T}_h)^{4N+2}$, such that

$$\mathcal{A}_5(\mathbf{\Upsilon_h}, \mathbf{\Phi_h}) = \mathcal{F}_5(\mathbf{\Phi_h}) \tag{35}$$

for all $(\mathbf{\Phi_h}) = (\Phi_0, \dots, \Phi_N) \in \mathcal{ND}_0(\mathcal{T}_h)^{4N+2}$, where the left-hand side $\mathcal{A}_5$ is given by

$$\mathcal{A}_5(\mathbf{\Upsilon_h}, \mathbf{\Phi_h}) := a(\mathbf{\Upsilon_y}, \mathbf{\Phi_v}) + b(\mathbf{\Phi_v}, \mathbf{\Upsilon_p}) + b(\mathbf{\Upsilon_y}, \mathbf{\Phi_q}) + c(\mathbf{\Upsilon_p}, \mathbf{\Phi_q})$$

with the bilinear forms

$$a(\mathbf{\Upsilon_y}, \mathbf{\Phi_v}) := \sum_{k=0}^{N} (1+\alpha)(\mathbf{y_k^c}, \mathbf{v_k^c})_{\mathbf{L_2}(\Omega)} + (\mathbf{y_k^s}, \mathbf{v_k^s})_{\mathbf{L_2}(\Omega)} + \alpha \sum_{k\neq l} (\mathbf{y_l^c}, \mathbf{v_k^c})_{\mathbf{L_2}(\Omega)},$$

$$c(\mathbf{\Upsilon_p}, \mathbf{\Phi_q}) := \frac{1}{\lambda} \sum_{j\in\{c,s\}} \sum_{k=0}^{N} (\mathbf{p_k^j}, \mathbf{q_k^j})_{\mathbf{L_2}(\Omega)},$$

$$b(\mathbf{\Upsilon_y}, \mathbf{\Phi_q}) := \sum_{k=0}^{N} \omega k \left[(\sigma \mathbf{y_k^s}, \mathbf{q_k^c})_{\mathbf{L_2}(\Omega)} - (\sigma \mathbf{y^c}, \mathbf{q_k^s})_{\mathbf{L_2}(\Omega)} \right] + \sum_{j\in\{c,s\}} (\nu\, \mathbf{curl}\, \mathbf{y_k^j}, \mathbf{curl}\, \mathbf{q_k^j})_{\mathbf{L_2}(\Omega)}.$$

Therein we used the following notations

$$\mathbf{\Upsilon_y} = (\mathbf{y_0^c}, \mathbf{y_1^c}, \mathbf{y_1^s}, \dots, \mathbf{y_N^c}, \mathbf{y_N^s}) \quad \text{and} \quad \mathbf{\Upsilon_p} = (\mathbf{p_0^c}, \mathbf{p_1^c}, \mathbf{p_1^s}, \dots, \mathbf{p_N^c}, \mathbf{p_N^s}),$$
$$\mathbf{\Phi_v} = (\mathbf{v_0^c}, \mathbf{v_1^c}, \mathbf{v_1^s}, \dots, \mathbf{v_N^c}, \mathbf{u_N^s}) \quad \text{and} \quad \mathbf{\Phi_q} = (\mathbf{q_0^c}, \mathbf{q_1^c}, \mathbf{q_1^s}, \dots, \mathbf{q_N^c}, \mathbf{p_N^s}).$$

The right-hand side $\mathcal{F}_5$ is given by

$$\mathcal{F}_5(\mathbf{\Phi_h}) := \sum_{k=0}^{N} (\mathbf{y_{d,k}^c}, \mathbf{v_k^c})_{\mathbf{L_2}(\Omega)} + (\mathbf{y_{d,k}^s}, \mathbf{v_k^s})_{\mathbf{L_2}(\Omega)} + \alpha(\mathbf{y_T}, \mathbf{v_k^c})_{\mathbf{L_2}(\Omega)}.$$

Furthermore we use the norm in the product space. In principal, we reuse the definition of the norm (19) for each mode k, but for technical reasons, we define a different splitting by

$$\|\mathbf{\Upsilon_h}\|_{\mathcal{C}_N}^2 = \|\mathbf{\Upsilon_y}\|_{\mathcal{C}_{N,1}}^2 + \|\mathbf{\Upsilon_p}\|_{\mathcal{C}_{N,2}}^2,$$

where the two components are given by

$$\|\mathbf{\Upsilon_y}\|_{\mathcal{C}_{N,1}}^2 = \sum_{k=0}^{N} \|(\mathbf{y_k^c}, \mathbf{y_k^s})\|_{\mathcal{C}_1}^2 \quad \text{and} \quad \|\mathbf{\Upsilon_p}\|_{\mathcal{C}_{N,2}}^2 = \sum_{k=0}^{N} \|(\mathbf{y_k^c}, \mathbf{y_k^s})\|_{\mathcal{C}_2}^2.$$

Here we use $\|(\cdot,\cdot)\|_{\mathcal{C}_1}$ and $\|(\cdot,\cdot)\|_{\mathcal{C}_2}$ as defined in (18). The main result is summarized in the following Lemma, that claims that an inf-sup condition and a sup-sup condition are fulfilled, where the constants are independent of h, N, ω, σ, ν and λ.

Lemma 8. *We have*

$$\underline{c}_5 \|(\mathbf{\Upsilon_h})\|_{\mathcal{C}_N} \leq \sup_{\mathbf{\Phi_h}\neq \mathbf{0}} \frac{\mathcal{A}_5(\mathbf{\Upsilon_h}, \mathbf{\Phi_h})}{\|\mathbf{\Phi_h}\|_{\mathcal{C}_N}} \leq \overline{c}_5 \|\mathbf{\Upsilon_h}\|_{\mathcal{C}_N}$$

for all $\mathbf{\Upsilon_h} \in \mathcal{ND}_0(\mathcal{T}_h)^{4N+2}$. *Here the constants* $\underline{c}_5$, $\overline{c}_5$ *are independent of* h, ω, ν, σ *and* λ.

Proof. In order to show the inf-sup and sup-sup condition for $\mathcal{A}_5$, we use Theorem 1. By definition of $c(\cdot,\cdot)$, we have

$$c(\mathbf{\Upsilon_p}, \mathbf{\Upsilon_p}) = \frac{1}{\lambda} \sum_{j\in\{c,s\}} \sum_{k=0}^{N} \|\mathbf{p_k^j}\|_{\mathbf{L_2}(\Omega)}^2.$$

Furthermore, using Cauchy's inequality and the definition of $\|\cdot\|_{\mathcal{C}_{N,2}}$, we obtain

$$\sup_{\mathbf{\Phi_q}\neq 0} \frac{b(\mathbf{\Upsilon_y},\mathbf{\Phi_q})}{\|\mathbf{\Phi_q}\|_{\mathcal{C}_{N,2}}} \leq \sqrt{\lambda} \sum_{j\in\{c,s\}} \sum_{k=0}^{N} \omega k(\sigma \mathbf{y_k^j}, \mathbf{y_k^j})_{\mathbf{L_2}(\Omega)} + (\nu\, \mathbf{curl}\, \mathbf{y_k^j}, \mathbf{curl}\, \mathbf{y_k^j})_{\mathbf{L_2}(\Omega)}.$$

It remains to derive the inf-sup condition for $b(\cdot,\cdot)$. Using the idea of Lemma 3, we can derive

$$\sup_{\mathbf{\Phi_q}\neq 0} \frac{b(\mathbf{\Upsilon_y},\mathbf{\Phi_q})}{\|\mathbf{\Phi_q}\|_{\mathcal{C}_{N,2}}} \geq \frac{1}{\sqrt{2}} \frac{\sum_{j\in\{c,s\}} \sum_{k=0}^{N} \omega k(\sigma \mathbf{y_k^j}, \mathbf{y_p^j})_{\mathbf{L_2}(\Omega)} + (\nu\, \mathbf{curl}\, \mathbf{y_k^j}, \mathbf{curl}\, \mathbf{y_k^j})_{\mathbf{L_2}(\Omega)}}{\|\mathbf{\Upsilon_y}\|_{\mathcal{C}_{N,2}}}.$$

Since $b(\cdot,\cdot)$ is skew symmetric, the same estimate can be obtained for the adjoint setting. Finally, it remains to estimate $a(\cdot,\cdot)$. Since we have

$$\sum_{k=0}^{N} \left[\|\mathbf{y_k^c}\|^2_{\mathbf{L_2}(\Omega)} + \sum_{l\neq k} (\mathbf{y_l^c}, \mathbf{y_k^c})_{\mathbf{L_2}(\Omega)} \right] = \sum_{l,k=0}^{N} (\mathbf{y_l^c}, \mathbf{y_k^c})_{\mathbf{L_2}(\Omega)} = (\sum_{l=0}^{N} \mathbf{y_l^c}, \sum_{k=0}^{N} \mathbf{y_k^c})_{\mathbf{L_2}(\Omega)} \geq 0,$$

a lower bound follows. The upper bound can be derived by applying Cauchy's inequality several times:

$$\sum_{j\in\{c,s\}} \sum_{k=0}^{N} \|\mathbf{y_k^j}\|^2_{\mathbf{L_2}(\Omega)} \leq a(\mathbf{\Upsilon_y},\mathbf{\Upsilon_y}) \leq (1+\alpha)N \sum_{j\in\{c,s\}} \sum_{k=0}^{N} \|\mathbf{y_k^j}\|^2_{\mathbf{L_2}(\Omega)}.$$

Combining the estimates according to Theorem 1 yields the desired result. □

7. MinRes convergence rates and implementation

From Lemma 1, Lemma 4, Lemma 5, Lemma 6, Lemma 7 and Lemma 8, we immediately obtain, that the spectral condition number of the preconditioned systems can be estimated by constants.

Proposition 1. *We have the condition number bounds*

$$\begin{aligned}
\kappa(\mathcal{P}^{-1}\mathcal{A}) &\leq (1+\sqrt{5})\sqrt{3},\\
\kappa(\mathcal{P}^{-1}\mathcal{A}_j) &\leq c_j \neq c_j(h,\omega,N), \quad j=1,2,\\
\kappa(\mathcal{P}^{-1}\mathcal{A}_3) &\leq c_3 \neq c_3(h,\omega,N,\mathcal{E}),\\
\kappa(\mathcal{C}^{-1}\mathcal{A}_4) &\leq c_4 \neq c_4(h,\omega,N,\nu,\sigma,\lambda,\mathcal{E}),\\
\kappa(\mathcal{C}_N^{-1}\mathcal{A}_5) &\leq c_5 \neq c_5(h,\omega,\nu,\sigma,\lambda).
\end{aligned}$$

The condition number estimates of the preconditioned systems immediately yield the convergence rate estimate of the MinRes method (see e.g. [11]). Therefore the number of MinRes iterations required for reducing the initial error by some fixed factor is independent of the space and time discretization parameters h and N, and, depending of the setting, also of the involved model parameters ν, σ and λ.

Still we have to solve the discrete variational problems connected with the norms $\mathcal{C}$ and $\mathcal{P}$. The solution of the variational problems connected with $\|\cdot\|_{\mathcal{C}}$ or $\|\cdot\|_{\mathcal{P}}$ supply good preconditioners for the variational problem associated with the bilinear form A_i, $i=1,2,3,4,5$. In large-scale computations, the individual parts of the norm and/or preconditioner have to be replaced by easy "invertible" and robust symmetric and positive definite norms and/or preconditioners. The application of the $\mathcal{C}$ preconditioner requires the solution of four variational problems of the form

$$(\nu\, \mathbf{curl}\, \mathbf{y_h}, \mathbf{curl}\, \mathbf{v_h})_{\mathbf{L_2}(\Omega_1)} + \omega(\sigma \mathbf{y_h}, \mathbf{v_h})_{\mathbf{L_2}(\Omega_1)} + \frac{1}{\sqrt{\lambda}} (\mathbf{y_h}, \mathbf{v_h})_{\mathbf{L_2}(\Omega_1)} = (\mathbf{f}, \mathbf{v_h})_{\mathbf{L_2}(\Omega_1)}.$$

Depending on the parameter setting $(\nu,\sigma,\omega,\lambda)$ candidates for robust and (almost) optimal preconditioners or solvers are multigrid preconditioners [1, 15], auxiliary space preconditioners [16, 32] and domain decomposition preconditioners [29, 17, 30]. The application of the $\mathcal{P}$ preconditioner additionally requires the solution of the variational problem corresponding to the norm

$$\|q_h\|_{\mathcal{P}_1} = \sup_{\mathbf{v_h} \in \mathcal{ND}_0(\mathcal{T}_h)} \frac{(\frac{1}{\sqrt{\lambda}} + k\omega)(\sigma \mathbf{v_h}, \nabla q_h)_{\mathbf{L_2}(\Omega)}}{\|\mathbf{v_h}\|_{\mathcal{C}_1}}.$$

Unfortunately, an evaluation of this norm is not straight forward, but we can use the following idea: By scaling the weakly divergence free part with $(\frac{1}{\sqrt{\lambda}} + k\omega)$ instead of $k\omega$ in Section 2, the same analysis can be developed and additionally, we can show the following norm equivalence:

$$\frac{1}{\min(1, \sigma_{\max})} \|q_h\|^2_{\mathcal{P}_1} \leq (\frac{1}{\sqrt{\lambda}} + k\omega)(\sigma \nabla q_h, \nabla q_h)_{\mathbf{L_2}(\Omega)} \leq \frac{1}{\min(1, \sigma_{\min})} \|q_h\|^2_{\mathcal{P}_1}.$$

Hence, we can use standard H^1 solvers for the evaluation of the Schur-complement and can still ensure convergence rates independent of h, N and ω. Note, that in the cases where the $\mathcal{P}$ preconditioner is required, we do not have robustness with respect to the remaining parameters anyway. The application of this preconditioner requires the solution of variational problems of the form

$$(\frac{1}{\sqrt{\lambda}} + k\omega)(\sigma \nabla p_h, \nabla q_h)_{\mathbf{L_2}(\Omega)} = (f_h, q_h)_{L_2(\Omega)}.$$

Depending on the parameter setting, candidates for robust and (almost) optimal preconditioners are multigrid or multilevel preconditioners [6, 26, 24], domain decomposition preconditioners [30] and auxiliary space preconditioners [25, 9].

8. Conclusion and Outlook

Summary and conclusion. The method developed in this work shows great potential for solving optimal control problems for multiharmonic eddy current problems in an efficient and optimal way. The key points of our method are the usage of a non-standard time discretization technique in terms of a truncated Fourier series and the construction of parameter-independent solvers for the resulting system of equations in the frequency domain. The theory developed in this paper establishes a theoretical estimate of the convergence rate of MinRes as a solver when our proposed preconditioner is applied. In the following table, we summarize the robustness results. Note that (σ) denotes robustness for $\sigma = const$.

setting	robust parameters
1) different control and observation domains	h, ω, N, (σ), Ω_1, Ω_2
2) desired curl state	h, ω, N, (σ)
3) desired final state	h, ω, σ, ν, λ
4) control constraints	h, ω, N, (σ), active index sets
5) state constraints	h, ω, N, σ, ν, λ, active index sets
distributed control	h, ω, N, σ, ν, λ,

Due to the natural decoupling of the frequency domain equations, an efficient parallel implementation of the solution procedure is straight-forward.

Outlook. The efficient treatment of full inequality constraints for the control and the state is much harder. The inclusion of such inequality constraints into the cost functional as a penalty term is one technique to handle this problem. However, this makes the optimal control problem nonlinear. In the case of nonlinear problems caused by the penalty technique or other nonlinearities in the state equation, we lose the decoupling of the optimality system for the Fourier coefficients. However, the block-diagonal preconditioners constructed for the linear case should also work in some nonlinear cases [5].

Up to now the analysis presented in this paper is limited to the case of uniformly positive conductivity σ. In typical applications the computational domain consists of conducting parts ($\sigma > 0$) and non-conducting parts ($\sigma = 0$). The non-conducting parts can be taken into account by performing a symmetrical FEM/BEM coupling method as done in [22] for the forward problem and in [20] for the optimal control problem. For the resulting system of FEM/BEM equations, a parameter-robust block-diagonal preconditioner can be constructed in the same manner.

The application of our solver to practical problems, including numerical validation, will be presented in a subsequent paper.

Acknowledgment

I want to thank Ulrich Langer and Markus Kollmann for fruitful and enlightening discussions.

References

[1] D. N. Arnold, R. S. Falk, and R. Winther. Multigrid in $H(\text{div})$ and $H(\text{curl})$. *Numer. Math.*, 85(2):197–217, 2000.

[2] L. Arnold and B. Harrach. A unified variational formulation for the parabolic-elliptic eddy current equations. Technical report, Fakultät für Mathematik, Technische Universität München, 2011.

[3] I. Babuška. Error-bounds for finite element method. *Numer. Math.*, 16(4):322–333, 1971.

[4] F. Bachinger, M. Kaltenbacher, and S. Reitzinger. An Efficient Solution Strategy for the HBFE Method. In *Proceedings of the IGTE '02 Symposium Graz, Austria*, pages 385–389, 2002.

[5] F. Bachinger, U. Langer, and J. Schöberl. Numerical analysis of nonlinear multiharmonic eddy current problems. *Numer. Math.*, 100(4):593–616, 2005.

[6] J. H. Bramble, J. E. Pasciak, and P. S. Vassilevski. Computational scales of Sobolev norms with application to preconditioning. *Math. Comp.*, 69(230):463–480, 2000.

[7] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[8] D. Copeland, M. Kolmbauer, and U. Langer. Domain decomposition solvers for frequency-domain finite element equation. In *Domain Decomposition Methods in Science and Engineering XIX*, volume 78 of *LNCSE*, pages 301–308, Heidelberg, 2011. Springer.

[9] R. D. Falgout and P. S. Vassilevski. On generalizing the algebraic multigrid framework. *SIAM J. Numer. Anal.*, 42(4):1669–1693 (electronic), 2004.

[10] H. D. Gersem., H. V. Sande, and K. Hameyer. Strong coupled multi-harmonic finite element simulation package. *COMPEL*, 20:535 –546, 2001.

[11] A. Greenbaum. *Iterative methods for solving linear systems*, volume 17 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.

[12] J. Gyselinck, P. Dular, C. Geuzaine, and W. Legros. Harmonic-balance finite-element modeling of electromagnetic devices: a novel approach. *Magnetics, IEEE Transactions on*, 38(2):521 –524, mar. 2002.

[13] R. Herzog and E. Sachs. Preconditioned conjugate gradient method for optimal control problems with control and state constraints. *SIAM J. Matrix Anal. Appl.*, 31(5):2291–2317, 2010.

[14] M. Hintermüller, K. Ito, and K. Kunisch. The primal-dual active set strategy as a semismooth Newton method. *SIAM J. Optim.*, 13(3):865–888 (electronic) (2003), 2002.

[15] R. Hiptmair. Multigrid method for Maxwell's equations. *SIAM J. Numer. Anal.*, 36(1):204–225 (electronic), 1999.

[16] R. Hiptmair and J. Xu. Nodal auxiliary space preconditioning in $\mathbf{H}(\mathbf{curl})$ and $\mathbf{H}(\text{div})$ spaces. *SIAM J. Numer. Anal.*, 45(6):2483–2509 (electronic), 2007.

[17] Q. Hu and J. Zou. A nonoverlapping domain decomposition method for Maxwell's equations in three dimensions. *SIAM J. Numer. Anal.*, 41(5):1682–1708 (electronic), 2003.

[18] M. Kollmann and M. Kolmbauer. A preconditioned minres solver for time-periodic parabolic optimal control problems. DK-Report 2011-10, Doctoral Program Computational Mathematics, Linz, August 2011.

[19] M. Kollmann, M. Kolmbauer, M. Kowalska, and U. Langer. Multiharmonic finite element analysis of a time-periodic parabolic optimal control problem. Technical report. in preparation.

[20] M. Kolmbauer. A robust fem-bem minres solver for distributed multiharmonic eddy current optimal control problems in unbounded domains. NuMa-Report 2011-08, Institute of Computational Mathematics, Linz, November 2011.

[21] M. Kolmbauer and U. Langer. A frequency-robust solver for the time-harmonic eddy current problem. In *Scientific Computing in Electrical Engineering SCEE 2010*, 2011. (accepted).

[22] M. Kolmbauer and U. Langer. A robust fem-bem solver for time-harmonic eddy current problems. NuMa-Report 2011-05, Institute of Computational Mathematics, Linz, May 2011.

[23] M. Kolmbauer and U. Langer. A robust preconditioned-minres-solver for distributed time-periodic eddy current optimal control problems. NuMa-Report 2011-04, Institute of Computational Mathematics, Linz, May 2011.

[24] J. Kraus. Additive schur complement approximation and application to multilevel preconditioning. RICAM Report 2011-22, Johann Radon Institute for Computational and Applied Mathematics (RICAM) of the Austrian Academy of Sciences, Linz, August 2011.

[25] J. K. Kraus. Algebraic multilevel preconditioning of finite element matrices using local Schur complements. *Numer. Linear Algebra Appl.*, 13(1):49–70, 2006.

[26] M. A. Olshanskii and A. Reusken. On the convergence of a multigrid method for linear reaction-diffusion problems. *Computing*, 65(3):193–202, 2000.

[27] C. C. Paige and M. A. Saunders. Solutions of sparse indefinite systems of linear equations. *SIAM J. Numer. Anal.*, 12(4):617–629, 1975.

[28] G. Paoli, O. Biro, and G. Buchgraber. Complex representation in nonlinear time harmonic eddy current problems. *Magnetics, IEEE Transactions on*, 34(5):2625 –2628, Sep. 1998.

[29] A. Toselli. Overlapping Schwarz methods for Maxwell's equations in three dimensions. *Numer. Math.*, 86(4):733–752, 2000.

[30] A. Toselli and O. Widlund. *Domain decomposition methods—algorithms and theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2005.

[31] F. Tröltzsch. *Optimal control of partial differential equations*, volume 112 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. Theory, methods and applications, Translated from the 2005 German original by Jürgen Sprekels.

[32] J. Xu and Y. Zhu. Robust preconditioner for H(curl) interface problems. In *Domain Decomposition Methods in Science and Engineering XIX*, volume 78 of *LNCSE*, pages 173–180, Heidelberg, 2011. Springer.

[33] S. Yamada and K. Bessho. Harmonic field calculation by the combination of finite element analysis and harmonic balance method. *Magnetics, IEEE Transactions on*, 24(6):2588 –2590, nov 1988.

[34] I. Yousept. Optimal control of maxwell's equations with regularized state constraints. *Computational Optimization and Applications*, pages 1–23. 10.1007/s10589-011-9422-2.

[35] W. Zulehner. Nonstandard norms and robust estimates for saddle point problems. *SIAM J. Matrix Anal. Appl.*, 32:536 – 560, 2011.

(M. Kolmbauer) Institute of Computational Mathematics, JKU Linz
E-mail address: `kolmbauer@numa.uni-linz.ac.at`

Latest Reports in this series

2009

[..]

2010

[..]		
2010-06	Helmut Gfrerer *Second Order Conditions for Metric Subregularity of Smooth Constraint Systems*	September 2010
2010-07	Walter Zulehner *Non-standard Norms and Robust Estimates for Saddle Point Problems*	November 2010
2010-08	Clemens Hofreither *L_2 Error Estimates for a Nonstandard Finite Element Method on Polyhedral Meshes*	December 2010
2010-09	Michael Kolmbauer and Ulrich Langer *A frequency-robust solver for the time-harmonic eddy current problem*	December 2010
2010-10	Clemens Pechstein and Robert Scheichl *Weighted Poincaré inequalities*	December 2010

2011

2011-01	Huidong Yang and Walter Zulehner *Numerical Simulation of Fluid-Structure Interaction Problems on Hybrid Meshes with Algebraic Multigrid Methods*	February 2011
2011-02	Stefan Takacs and Walter Zulehner *Convergence Analysis of Multigrid Methods with Collective Point Smoothers for Optimal Control Problems*	February 2011
2011-03	Michael Kolmbauer *Existance and Uniqueness of Eddy Current Problems in Bounded and Unbounded Domains*	May 2011
2011-04	Michael Kolmbauer and Ulrich Langer *A Robust Preconditioned-MinRes-Solver for Distributed Time-Periodic Eddy Current Optimal Control*	May 2011
2011-05	Michael Kolmbauer and Ulrich Langer *A Robust FEM-BEM Solver for Time-Harmonic Eddy Current Problems*	May 2011
2011-06	Markus Kollmann and Michael Kolmbauer *A Preconditioned MinRes Solver for Time-Periodic Parabolic Optimal Control Problems*	August 2011
2011-07	Nicole Spillane, Victorita Dolean, Patrice Hauret, Frédérik Nataf, Clemens Pechstein and Robert Scheichl *Abstract Robust Coarse Spaces for Systems of PDEs via Generalized Eigenproblems in the Overlaps*	November 2011
2011-08	Michael Kolmbauer *A Robust FEM-BEM MinRes Solver for Distributed Multiharmonic Eddy Current Optimal Control Problems in Unbounded Domains*	November 2011
2011-09	Michael Kolmbauer *Efficient Solvers for Multiharmonic Eddy Current Optimal Control Problems with Various Constraints*	November 2011

From 1998 to 2008 reports were published by SFB013. Please see
`http://www.sfb013.uni-linz.ac.at/index.php?id=reports`
From 2004 on reports were also published by RICAM. Please see
`http://www.ricam.oeaw.ac.at/publications/list/`
For a complete list of NuMa reports see
`http://www.numa.uni-linz.ac.at/Publications/List/`