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Weighted Poincaré inequalities

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WEIGHTED POINCARÉ INEQUALITIES

CLEMENS PECHSTEIN[1*] AND ROBERT SCHEICHL[2]

Abstract. Poincaré type inequalities are a key tool in the analysis of partial differential equations. They play a particularly central role in the analysis of domain decomposition and multilevel iterative methods for second-order elliptic problems. When the diffusion coefficient varies within a subdomain or within a coarse grid element, then condition number bounds for these methods based on standard Poincaré inequalities may be overly pessimistic. In this paper we present new results on weighted Poincaré type inequalities for very general classes of coefficients that lead to sharper bounds independent of any possible large variation in the coefficients. The main requirement on the coefficients is some form of quasi-monotonicity which we will carefully describe and analyse. The Poincaré constants depend on the topology and the geometry of regions of relatively high and/or low coefficient values, and we will study these dependencies in detail. Applications of the inequalities in the analysis of the geometric multigrid, the two-level overlapping Schwarz and the FETI methods can be found in [25, 30].

1. Introduction

Poincaré type inequalities are a key tool in the analysis of partial differential equations (PDEs). They are at the heart of uniqueness results, of a priori and a posteriori error analyses of discretisation schemes, and of convergence analyses of iterative solution strategies, in particular in the analysis of domain decomposition (DD) and multigrid (MG) methods for finite element (FE) discretisations of elliptic PDEs of the type

$$-\nabla \cdot (\alpha \, \nabla u) = f \, . \tag{1.1}$$

In many applications, such as porous media flow or electrostatics, the coefficient function $\alpha = \alpha(x)$ in (1.1) is discontinuous and varies over several orders of magnitude throughout the domain in a possibly very complicated way. Standard analyses of multilevel iterative methods for (1.1) that use classical Poincaré type inequalities will often lead to pessimistic bounds in this case. If the subdomain partition in a DD method or the coarsest grid in a MG method can be chosen such that $\alpha(x)$ is constant (or almost constant) on each subdomain or on each coarse grid element, then it is possible to prove bounds that are independent of the coefficient variation (cf. [8, 17, 34, 37]). However, if this is not possible and the coefficient varies strongly within a subdomain or within a coarse grid element, then the classical bounds depend on the local variation of the coefficient, which may be overly pessimistic in many cases. To obtain sharper bounds in some of these cases, it is possible to refine the standard analyses and use Poincaré inequalities on annulus type boundary layers of each subdomain [13, 24, 26, 29], or weighted Poincaré type inequalities [11, 25, 30]. See also [7, 9, 12, 14, 16, 22, 28, 39] for related work.

Let D be a bounded Lipschitz domain in $\mathbb{R}^d$ where $d \in \{1, 2, 3\}$. Throughout the paper we consider coefficients or weight functions α with

$$\alpha \;\in\; L^\infty_+(D) \;:=\; \big\{\alpha \in L^\infty(D) : \inf_{x \in D} \alpha(x) > 0\big\}. \tag{1.2}$$

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Such a weight function induces the weighted norm and seminorm

$$
(1.3)\qquad
\begin{aligned}
\|u\|_{L^2(D),\alpha} &:= \Big(\int_D \alpha(x)\, |u(x)|^2 \, dx \Big)^{1/2}, \\
|u|_{H^1(D),\alpha} &:= \Big(\int_D \alpha(x)\, |\nabla u(x)|^2 \, dx \Big)^{1/2}.
\end{aligned}
$$

We are interested in finding bounds for the constant $C_{P,\alpha}(D)$ in the weighted Poincaré type inequality

$$
(1.4)\qquad \inf_{c\in\mathbb{R}} \|u-c\|^2_{L^2(D),\alpha} \;\le\; C_{P,\alpha}(D)\, \mathrm{diam}(D)^2\, |u|^2_{H^1(D),\alpha} \qquad \forall u \in H^1(D).
$$

that are independent of the values that the weight function α takes on D.

Clearly, $C_{P,\alpha}(D)$ depends on the shape of the domain D. However, one easily shows by dilation that $C_{P,\alpha}(D)$ is independent of $\mathrm{diam}(D)$. The infimum in (1.4) is attained when choosing the constant

$$
(1.5)\qquad c \;=\; \overline{u}^{D,\alpha} \;:=\; \frac{\int_D \alpha\, u\, dx}{\int_D \alpha\, dx},
$$

which is the α-weighted average of u over D (cf. e.g. [5], [11, Lemma 4]). This is easily seen from a variational argument. The functional on the left hand side of (1.4) is convex with respect to c, and hence the infimum is attained if and only if

$$
0 \;=\; \frac{d}{dc} \int_D \alpha\, |u-c|^2 \, dx \;=\; -2 \int_D \alpha\, (u-c)\, dx.
$$

If $\mathrm{diam}(D) = 1$, the best constant $C_{P,\alpha}(D)$ is the inverse of the second smallest eigenvalue of the generalised eigenvalue problem

$$
(1.6)\qquad -\nabla\cdot(\alpha\,\nabla u) \;=\; \lambda\,\alpha\, u \qquad \text{in } D,
$$

$$
(1.7)\qquad \alpha\,\nabla u\cdot n \;=\; 0 \qquad \text{on } \partial D,
$$

see, for example [12]. For general weight functions α, we can obtain a bound for $C_{P,\alpha}(D)$ in (1.4) from the usual Poincaré inequality. Let $\overline{u}^D := \overline{u}^{D,1}$ be the usual average (cf. (1.5)). Then, it is easily shown that

$$
\|u-\overline{u}^D\|^2_{L^2(D),\alpha} \;\le\; \sup_{x,y\in D} \frac{\alpha(x)}{\alpha(y)}\, C_P(D)\, \mathrm{diam}(D)^2\, |u|^2_{H^1(D),\alpha}.
$$

where $C_P(D) = C_{P,1}(D)$ is the usual Poincaré constant on D. Thus, this bound for $C_{P,\alpha}(D)$ depends on the *global variation* $\sup_{x,y\in D} \frac{\alpha(x)}{\alpha(y)}$, and if α is highly variable, this may be very large and very pessimistic.

We note that although weighted Poincaré inequalities have been investigated a lot in the literature, estimates of the Poincaré constant $C_{P,\alpha}$ that show certain robustness in α are hardly known. Chua [4] showed that the weighted Poincaré inequality holds for domains satisfying the Boman chain condition with weights α from a Muckenhoupt class (i.e., α and α^{-1} are locally in some Lebesgue space, see [21]). Chua's paper is based on the early work by Iwaniec and Nolder [15], see also [10, 20] for related work. The constant in the Poincaré inequality depends in general on the weight. A similar result is obtained by Zhikov [38] for weights $\alpha \in L^r$ with $\alpha^{-1} \in L^s$ with $2\,d^{-1} = r^{-1} + s^{-1}$. Also there, the Poincaré constant depends on α. In [5], Chua and Wheeden provide explicit estimates for the Poincaré constant for the class of convex domains Ω with weights α that are a positive power of a non-negative concave function. Note that concavity implies *continuity*. Recently, Veeser and Verfürth [36] refined these results to star shaped domains, where the weight function satisfies a certain concavity property with respect to the central point of the star (see Condition (2.3) in [36] for more details, and see [35] on how to use these inequalities in (explicit) a-posteriori error estimation). To the best of our knowledge, the first paper that deals with robust estimates of the weighted Poincaré constant for *discontinuous* weight functions is [11]. There, Efendiev and Galvis show that for piecewise constant coefficients α, if the largest value is attained in a connected region Ω_1 and if all the other regions of constant α are inclusions of (or at least bordering) Ω_1, then $C_{P,\alpha}$ is independent of the *values* of α, in particular of possibly high *contrast*.

In the present paper we want to collect and expand on the results in [11, 25, 27] and present sharp constants for weighted Poincaré-type inequalities that are independent of the value of the weight function for a rather general class of coefficients. In Section 2.1, we will define a class of *quasi-monotone* piecewise constant weight functions (far more general than in [11]) for which we can make $C_{P,\alpha}(D)$ totally independent of the *values* of α. To get bounds for $C_{P,\alpha}(D)$ in (1.4), we will choose averages over certain manifolds rather than over D. In Section 2.2 we will achieve similar results for an even more general class of non-constant coefficients. In many applications, especially in the analysis of MG and DD methods, Poincaré type inequalities are not needed on all of $H^1(D)$ but only for the subset of finite element functions. This restriction allows for a larger class of coefficients α, where we can show discrete analogues of inequality (1.4). This issue will be treated in Section 3. Even if the Poincaré constant $C_{P,\alpha}(D)$ can be bounded independent of the values of α, it will in general depend on the topology and geometry of the partition of D underlying the piecewise constant weight function. In Section 4, we will work out what this *geometric* dependence looks like. Since this issue can be rather complicated in two and three space dimensions, we present a series of general technical tools and analyse a few exemplary cases in detail.

Extensions to PDEs/inequalities where α is replaced by an isotropic tensor are straightforward, whereas the case of anisotropic tensors is substantially harder.

Applications of these novel weighted Poincaré–type inequalities in the analysis of geometric multigrid, as well as of two-level overlapping Schwarz and FETI domain decomposition methods can be found in [25, 30].

2. Weighted Poincaré type inequalities in H^1

Let us start by considering inequalities for piecewise constant weight functions (Section 2.1). We will return to more general weight functions in Section 2.2.

2.1. Quasi-monotone piecewise constant weight functions. Let the weight function $\alpha \in L^\infty_+(D)$ be piecewise constant with respect to a non-overlapping partitioning of D into open, connected Lipschitz polygons (polyhedra) $\mathcal{Y} := \{Y_\ell : \ell = 1, \dots, n\}$, i. e.

$$\overline{D} = \bigcup_{\ell=1}^{n} \overline{Y}_\ell \qquad \text{and} \qquad \alpha|_{Y_\ell} \equiv \alpha_\ell \tag{2.1}$$

for some constants α_ℓ. We will drop this condition in Section 2.2.

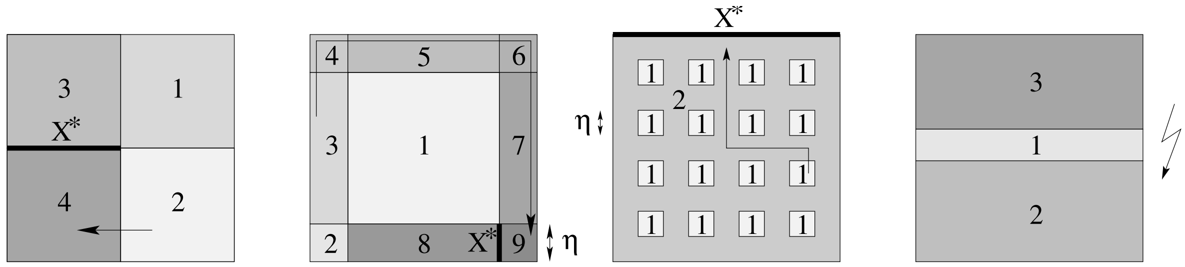
To simplify the presentation we set $H := \mathrm{diam}(D)$ and define for any $u \in H^1(D)$ and for any $(d-1)$–dimensional manifold $X \subset \overline{D}$ the average

$$\overline{u}^X := \begin{cases} \frac{1}{\mathrm{meas}_{d-1}(X)} \int_X u\, ds, & \text{if } d > 1, \\ \frac{1}{\mathrm{meas}_{d-1}(X)} \sum_{x \in X} u(x) & \text{if } d = 1, \qquad \text{where } \ \mathrm{meas}_0(X) := \sum_{x\in X} 1. \end{cases}$$

Definition 2.1. Suppose $\alpha \in L^\infty_+(D)$ satisfies (2.1) and $\ell^* := \mathrm{argmax}\{\alpha_\ell\}_{\ell=1}^n$.

(a) We call the region $P_{\ell_1,\ell_s} := (\overline{Y}_{\ell_1} \cup \overline{Y}_{\ell_2} \cup \cdots \cup \overline{Y}_{\ell_s})^\circ$, $1 \le \ell_1, \dots, \ell_s \le n$, a *quasi-monotone path* from Y_{ℓ_1} to Y_{ℓ_s} (with respect to α), if the following two conditions are satisfied:
   (i) for each $i = 1, \dots, s-1$, the regions $\overline{Y}_{\ell_i}$ and $\overline{Y}_{\ell_{i+1}}$ share a common $(d-1)$-dimensional manifold X_i,
   (ii) $\alpha_{\ell_1} \le \alpha_{\ell_2} \le \cdots \le \alpha_{\ell_s}$.

(b) We say that α is *quasi-monotone on D*, if for any $k = 1, \dots, n$ there exists a quasi-monotone path P_{k,ℓ^*} from Y_k to Y_{ℓ^*}. Let s_k denote the length of P_{k,ℓ^*}.



(a) (b) (c) (d)

FIGURE 1. The numbering of the regions Y_ℓ in these examples is according to the relative sizes of the weights α_ℓ on each region, with the smallest weight in region Y_1. Examples (a–c) are quasi-monotone in the sense of Definition 2.1. In each case a typical path and a suitable manifold X^* are displayed. Example (d) is not quasi-monotone.

(c) Let $X^* \subset \overline{Y}_{\ell^*}$ be a $(d-1)$–dimensional manifold. For each $k = 1, \ldots, n$, let $c_k^{X^*} > 0$ be the best constant such that

$$\|u - \overline{u}^{X^*}\|^2_{L^2(Y_k)} \;\le\; c_k^{X^*} H^2 \, |u|^2_{H^1(P_{k,\ell^*})} \quad \forall u \in H^1(P_{k,\ell^*}) \tag{2.2}$$

and set $C^*_{P,\alpha} := \sum_{k=1}^n c_k^{X^*}$.

Note that the constant $C^*_{P,\alpha}$ in Definition 2.1(c) depends on the choice of manifold $X^* \subset \overline{Y}_{\ell^*}$ and of the paths $\{P_{k,\ell^*}\}_{k=1}^n$. The above definition is a generalisation of the notion of quasi-monotone coefficients introduced in [8]. In Figure 1(a–c) we give some examples of weight functions that satisfy Definition 2.1. The coefficient shown in Figure 1(d) fails to be quasi-monotone.

The following theorem provides a weighted Poincaré inequality for quasi-monotone weight functions α. The constant in the inequality is $C^*_{P,\alpha}$ from Definition 2.1(c) which is clearly independent of the values that α takes on D.

Theorem 2.2 (weighted Poincaré inequality – piecewise constant case)**.** *Let $\alpha \in L^\infty_+(D)$ be quasi-monotone on D in the sense of Definition 2.1. Then*

$$\inf_{c \in \mathbb{R}} \|u - c\|^2_{L^2(D),\alpha} \;\le\; C^*_{P,\alpha} H^2 \, |u|^2_{H^1(D),\alpha} \quad \forall u \in H^1(D), \tag{2.3}$$

where $C^*_{P,\alpha}$ is the constant defined in Definition 2.1(c).

Proof. For simplicity, we assume that $H = \mathrm{diam}(D) = 1$. The general case follows from a dilation argument. We set $c = \overline{u}^{X^*}$ (where X^* is the manifold chosen in Definition 2.1) and assume without loss of generality that $\overline{u}^{X^*} = 0$. Otherwise we can set $\widehat{u} := u - \overline{u}^{X^*}$ and use the fact that $|\widehat{u}|_{H^1(D),\alpha} = |u|_{H^1(D),\alpha}$.

Let $k \in \{1, \ldots, n\}$ be fixed. Then, due to the assumption (2.1) on the weight function α, we have

$$\|u\|^2_{L^2(Y_k),\alpha} \;=\; \alpha_k \, \|u\|^2_{L^2(Y_k)} \, .$$

Combining this identity with inequality (2.2) and using the fact that the value of α is monotonically increasing in the path from Y_k to Y_{ℓ^*}, we obtain

$$\|u\|^2_{L^2(Y_k),\alpha} \;\le\; c_k^{X^*} \alpha_k \, |u|^2_{H^1(P_{k,\ell^*})} \;\le\; c_k^{X^*} \, |u|^2_{H^1(P_{k,\ell^*}),\alpha} \;\le\; c_k^{X^*} \, |u|^2_{H^1(D),\alpha} \, .$$

The proof is complete by adding up the above estimates for $k = 1, \ldots, n$. □

As we can see from the proof of Theorem 2.2, inequality (2.3) does not only hold for the infimum, i.e. for the weighted average $c = \overline{u}^{D,\alpha}$, but also for $c = \overline{u}^{X^*}$ where X^* may be any $(d-1)$–dimensional manifold in Y_{ℓ^*}.

Although the definition of the constant $C^*_{P,\alpha}$ in Definition 2.1(c) suggests that it grows with the number n of subregions, this is not the case in general. The reason is that on the left hand side in (2.2), the L^2-norm is taken only over Y_k and *not* over the whole path P_{k,ℓ^*}. We will discuss this issue extensively in Section 4. However, we would like to give already at this stage a general tool, Lemma 2.4 below, on how the inequalities (2.2) are related to more common Poincaré inequalities on each of the individual subregions Y_k.

Definition 2.3. For any bounded Lipschitz domain $Y \subset \mathbb{R}^d$, $d = 1, 2, 3$, and for any $(d-1)$–dimensional manifold $X \subset \overline{Y}$, let $C_P(Y; X) > 0$ denote the best constant such that the following Poincaré type inequality holds:

$$\|u - \overline{u}^X\|^2_{L^2(Y)} \;\le\; C_P(Y;X)\,\mathrm{diam}(Y)^2\,|u|^2_{H^1(Y)} \quad \forall u \in H^1(Y). \tag{2.4}$$

Lemma 2.4. *Suppose $\alpha \in L^\infty_+(D)$ is quasi-monotone and P_{k,ℓ^*} is any of the paths in Definition 2.1(b) with $\ell_1 = k$ and $\ell_s = \ell^*$. For convenience let $X_0 := X_1$ and $X_s := X^*$. Then the constant $c_k^{X^*}$ in Definition 2.1(c) can be bounded by*

$$c_k^{X^*} \;\le\; 4\sum_{i=1}^{s} \frac{\mathrm{meas}(Y_k)}{\mathrm{meas}(Y_{\ell_i})}\,\frac{\mathrm{diam}(Y_{\ell_i})^2}{H^2}\,\max\Big\{C_P(Y_{\ell_i};X_{i-1}), C_P(Y_{\ell_i};X_i)\Big\}.$$

Proof. By a telescoping argument we have

$$\|u - \overline{u}^{X^*}\|_{L^2(Y_k)} \;\le\; \|u - \overline{u}^{X_1}\|_{L^2(Y_k)} + \sum_{i=2}^{s}\sqrt{\mathrm{meas}(Y_k)}\,\big|\overline{u}^{X_{i-1}} - \overline{u}^{X_i}\big|. \tag{2.5}$$

Estimate (2.4) yields a bound for the first term on the right hand side, i.e.

$$\|u - \overline{u}^{X_1}\|^2_{L^2(Y_k)} \;\le\; C_P(Y_k;X_1)\,\mathrm{diam}(Y_k)^2\,|u|^2_{H^1(Y_k)}. \tag{2.6}$$

For i fixed, we can also conclude from inequality (2.4) that

$$\begin{aligned}\big|\overline{u}^{X_{i-1}} - \overline{u}^{X_i}\big|^2 \;&\le\; \frac{2}{\mathrm{meas}(Y_{\ell_i})}\Big(\|\overline{u}^{X_{i-1}} - u\|^2_{L^2(Y_{\ell_i})} + \|u - \overline{u}^{X_i}\|^2_{L^2(Y_{\ell_i})}\Big)\\ &\le\; 4\,\max\Big\{C_P(Y_{\ell_i};X_{i-1}), C_P(Y_{\ell_i};X_i)\Big\}\frac{\mathrm{diam}(Y_{\ell_i})^2}{\mathrm{meas}(Y_{\ell_i})}\,|u|^2_{H^1(Y_{\ell_i})} \qquad (2.7)\end{aligned}$$

(this is essentially a Bramble-Hilbert type argument). An application of Cauchy's inequality (in $\mathbb{R}^s$) yields the final result. ☐

Note that in one dimension, due to Lemma 2.4, the Poincaré constant $C^*_{P,\alpha}$ is $\mathcal{O}(1)$ as $n \to \infty$, as the following corollary shows. The situation in two and three dimensions is more complicated and is left until Section 4.

Corollary 2.5. *Let $d = 1$. If α is piecewise constant with respect to $\{Y_\ell\}_{\ell=1}^n$ and quasi-monotone in the sense of Definition 2.1, then $C^*_{P,\alpha} = \mathcal{O}(1)$ as $n \to \infty$.*

Proof. We assume w.l.o.g. that $D = (0, 1)$ and $X^* = 1$. (Note that in this case quasi-monotonicity in the sense of Definition 2.1 is equivalent to the usual monotonicity.) Let us assume that the regions Y_ℓ are numbered consecutively from left to right, and that $X_\ell := \overline{Y}_\ell \cap \overline{Y}_{\ell+1}$, for $\ell = 1, \ldots, n-1$, with $X_n := X^*$. It follows from the Fundamental Theorem of Calculus that

$$\|u - u(X_{\ell-1})\|^2_{L^2(Y_\ell)} \le \mathrm{diam}(Y_\ell)^{-2}|u|^2_{H^1(Y_\ell)} \quad \forall u \in H^1(Y_\ell) \quad \forall \ell = 1, \ldots, n. \tag{2.8}$$

Hence, $C_P(Y_\ell; X_{\ell-1}) \le 1$. The same is true, if we replace $X_{\ell-1}$ by X_ℓ. Since for $d = 1$ we have $\mathrm{meas}(Y_\ell) = \mathrm{diam}(Y_\ell)$, it follows from Lemma 2.4 that

$$c_k^{X^*} \;\le\; 4\,\mathrm{diam}(Y_k)\sum_{\ell=k}^{n}\mathrm{diam}(Y_\ell) \;\le\; 4\,\mathrm{diam}(Y_k) \quad \forall k = 1, \ldots, n,$$

and so $C^*_{P,\alpha} \le 4 = \mathcal{O}(1)$ as $n \to \infty$. ☐

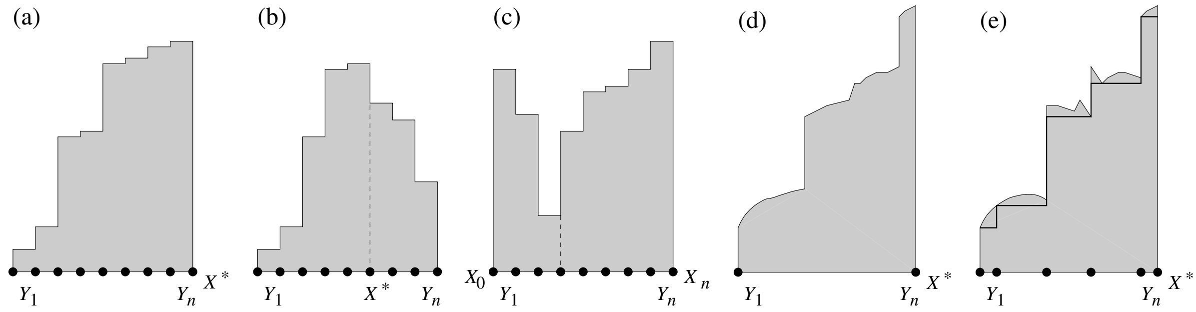


FIGURE 2. Examples of quasi-monotone weight functions in 1D. Cases (a–b) are quasi-monotone in the sense of Definition 2.1. Case (c) is Γ-quasi-monotone in the sense of Definition 2.6 with $\Gamma = \{X_0, X_n\}$, Cases (d–e) are quasi-monotone in the sense of Definition 2.8 (see Section 2.2 below).

Note that it was crucial to define $c_k^{X^*}$ as done in Definition 2.1. Using a standard Poincaré type inequality for P_{k,ℓ^*}, such as

$$\|u - \overline{u}^{X^*}\|^2_{L^2(P_{k,\ell^*})} \;\leq\; C_P(P_{k,\ell^*}; X^*)\, \mathrm{diam}(P_{k,\ell^*})^2\, |u|^2_{H^1(P_{k,\ell^*})}\,,$$

would lead to a very pessimistic bound for the Poincaré constant in (2.3):

$$C^*_{P,\alpha} \;\leq\; \sum_{k=1}^{n} C_P(P_{k,\ell^*}; X^*)\, \frac{\mathrm{diam}(P_{k,\ell^*})^2}{H^2}\,.$$

In our 1D example in Corollary 2.5 this would in general lead to $C^*_{P,\alpha} = \mathcal{O}(n)$.

An inequality similar to that in Theorem 2.2 holds if u vanishes on part of the boundary of D. This is sometimes referred to as a Friedrichs inequality.

Definition 2.6. Suppose $\alpha \in L^\infty_+(D)$ satisfies (2.1) and $\Gamma \subset \partial D$.

(a) We say that α is Γ-*quasi-monotone on* D, if for all $k = 1, \ldots, n$ there exists an index ℓ_k^* and a quasi-monotone path P_{k,ℓ_k^*} (with respect to α) from Y_k to $Y_{\ell_k^*}$, such that $\partial Y_{\ell_k^*} \cap \Gamma$ is a $(d-1)$–dimensional manifold.

(b) For each $k = 1, \ldots, n$, let $c_k^\Gamma > 0$ be the best constant such that

$$\|u\|^2_{L^2(Y_k)} \;\leq\; c_k^\Gamma H^2\, |u|^2_{H^1(P_{k,\ell_k^*})} \quad \forall u \in H^1(P_{k,\ell_k^*}),\; u|_\Gamma = 0. \tag{2.9}$$

and set $C^\Gamma_{F,\alpha} := \sum_{k=1}^n c_k^\Gamma$.

Again the constant $C^\Gamma_{F,\alpha}$ in Definition 2.6(b) is clearly independent of the actual values that α takes on D. A one-dimensional example of a Γ–quasi–monotone function is given in Figure 2(c). Note that this function is not quasi–monotone in the sense of Definition 2.1, while the example in Figure 2(b) is not Γ–quasi–monotone in the sense of Definition 2.6 for any choice of $\Gamma \subset \partial D$.

Theorem 2.7 (weighted Friedrichs inequality – piecewise constant case)**.** *Let* $\Gamma \subset \partial D$ *and suppose that* $\alpha \in L^\infty_+(D)$ *is* Γ*-quasi-monotone on* D *in the sense of Definition 2.6. Then*

$$\|u\|^2_{L^2(D),\alpha} \;\leq\; C^\Gamma_{F,\alpha} H^2\, |u|^2_{H^1(D),\alpha} \quad \textit{for all } u \in H^1(D) \textit{ with } u|_\Gamma = 0,$$

where $C^\Gamma_{F,\alpha}$ *is the constant defined in Definition 2.6(b).*

Proof. The proof is analogous to that of Theorem 2.2. □

For the remainder of this paper we will restict our attention to weighted Poincaré type inequalities (cf. Theorem 2.2), but we remark that there are always analogous statements for weighted Friedrichs type inequalities (cf. Theorem 2.7) that we will not mention or prove explicitly.

2.2. **General weight functions.** In this subsection we digress briefly to discuss more general non-constant weight functions. To do this we generalise our definition of quasi-monotonicity. Our bounds are then not completely independent of the values of α, but they will only depend on the *local variation.* Finally, we will show that our bounds are in a certain sense sharp.

Definition 2.8. Let $\mathcal{Y} := \{Y_\ell\}_{\ell=1}^n$ be a non-overlapping partition of D. A weight function $\alpha \in L^\infty_+(D)$ is called *(macroscopically) quasi-monotone* with respect to $\mathcal{Y}$ if the auxiliary piecewise constant weight function $\underline{\alpha} \in L^\infty_+(D)$ defined by

$$\underline{\alpha}(x) \ := \ \inf_{y\in Y_\ell} \alpha(y)\,, \qquad \text{for all } x \in Y_\ell\,,$$

is quasi-monotone on D in the sense of Definition 2.1. (For a typical example see Figure 2(e).)

Clearly, Definition 2.8 is a generalisation of Definition 2.1. Any $\alpha \in L^\infty_+(D)$ that satisfies (2.1) and is quasi-monotone in the sense of Definition 2.1 is also macroscopically quasi-monotone with respect to $\mathcal{Y}$ in the sense of Definition 2.8 with $\underline{\alpha} \equiv \alpha$. Moreover, any weight function $\alpha \in L^\infty_+(D)$ is macroscopically quasi-monotone in the sense of Definition 2.8 with respect to the trivial partition $\mathcal{Y} := \{D\}$. However, a finer partition may lead to a better bound for the Poincaré constant $C_{P,\alpha}$ in the following theorem (which is a generalisation of Theorem 2.2).

Analogously to $\underline{\alpha}$ let us also define $\overline{\alpha} \in L^\infty_+(D)$ such that

$$\overline{\alpha}(x) \ := \ \sup_{y\in Y_\ell} \alpha(y)\,, \qquad \text{for all } x \in Y_\ell\,.$$

Theorem 2.9. *(weighted Poincaré inequality – general case) Let $\mathcal{Y} := \{Y_\ell\}_{\ell=1}^n$ be a non-overlapping partition of D and let $\alpha \in L^\infty_+(D)$ be macroscopically quasi-monotone with respect to $\mathcal{Y}$ in the sense of Definition 2.8. Then*

$$\inf_{c\in\mathbb{R}} \|u-c\|^2_{L^2(D),\alpha} \ \le \ C^*_{P,\underline{\alpha}} \left\| \frac{\overline{\alpha}}{\underline{\alpha}} \right\|_{L^\infty(D)} H^2\, |u|^2_{H^1(D),\alpha} \quad \textit{for all } u \in H^1(D),$$

where $C^*_{P,\underline{\alpha}}$ is the constant in Definition 2.1(c) for the auxiliary function $\underline{\alpha}$.

Proof. We proceed as in the proof of Theorem 2.2 and assume without loss of generality that $\overline{u}^{X^*} = 0$ and $\mathrm{diam}(D) = 1$. Then, using again Theorem 2.2, inequality (2.2) and the quasi-monotonicity of $\underline{\alpha}$, we have

$$\begin{aligned}\|u\|^2_{L^2(Y_k),\alpha} \ &\le \ \sup_{x\in Y_k} \alpha(x)\, \|u\|^2_{L^2(Y_k)} \\ &\le \ \sup_{x\in Y_k} \alpha(x)\, c_k^{X^*}\, |u|^2_{H^1(P_{k,\ell^*})} \ \le \ \frac{\sup_{x\in Y_k}\alpha(x)}{\inf_{y\in Y_k}\alpha(y)}\, c_k^{X^*}\, |u|^2_{H^1(P_{k,\ell^*}),\underline{\alpha}}\,.\end{aligned}$$

Obviously, $|u|_{H^1(P_{\ell,k}),\underline{\alpha}} \le |u|_{H^1(P_{\ell,k}),\alpha}$, which completes the proof. □

Theorem 2.9 states that the Poincaré constant $C_{P,\alpha}$ depends only on the *local* variation of α on each of the subregions $Y_k \in \mathcal{Y}$. However, since we are free to choose the partition $\mathcal{Y}$, it is in principle possible to obtain a Poincaré constant that is completely independent of the variation of α (even for exponentially growing coefficients), by letting $n \to \infty$ – provided α remains macroscopically quasi-monotone w.r.t. $\mathcal{Y}$ as we let $n \to \infty$. We would like to illustrate this in one dimension. The following corollary follows immediately from Theorem 2.9 and the proof of Corollary 2.5.

Corollary 2.10. *Let $D = [0,1]$ and $X^* \in [0,1]$. If α is monotonically non-decreasing on $(0, X^*)$ and monotonically non-increasing on $(X^*, 1)$, then*

$$\inf_{c\in\mathbb{R}} \|u-c\|^2_{L^2(D),\alpha} \ \le \ 4\,|u|^2_{H^1(D),\alpha} \quad \forall u \in H^1(D).$$

Theorem 2.9 also shows that we still get good bounds for $C_{P,\alpha}$, even if we do not have *strict* quasi-monotonicity (in the sense of Definition 2.1). An example of this is the case in Figure 1(d) with $\alpha_1 = 1$, $\alpha_2 = 10$ and $\alpha_3 \gg 10$. Applying Theorem 2.9 with the partition $\mathcal{Y} := \{Y_1 \cup Y_2, Y_3\}$

(instead of Theorem 2.2), the maximum local variation is $\|\overline{\alpha}/\underline{\alpha}\|_{L^\infty(D)} = 10$ and so it follows from Theorem 2.9 that $C_{P,\alpha} = O(1)$ as $\alpha_3 \to \infty$.

However, the bound in Theorem 2.9 deteriorates when quasi-monotonicity is strongly violated. For the example in Figure 1(d) it can be shown that

$$C_{P,\alpha} \;\geq\; c \min\left\{\frac{\alpha_2}{\alpha_1}, \frac{\alpha_3}{\alpha_1}\right\}$$

(cf. [25, Sect. 3.3]). The next lemma shows that quasi-monotonicity is in fact a necessary condition for $C_{P,\alpha}$ to remain bounded when the contrast in the coefficient goes to infinity.

Proposition 2.11. *Suppose that $\alpha \in L^\infty_+(D)$ satisfies (2.1) and the subregions $\{Y_\ell\}_{\ell=1}^n$ are ordered such that $\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_1$. If α is not quasi-monotone in the sense of Definition 2.1, then there exist indices k, j with $n > k > j \geq 1$ and a constant $C > 0$ independent of $\{\alpha_\ell\}_{\ell=1}^n$ such that*

$$\alpha_k \;>\; \alpha_j \quad \textit{and} \quad C_{P,\alpha} \;\geq\; C\,\frac{\alpha_k}{\alpha_j}\,,$$

i.e. $C_{P,\alpha} \to \infty$ as $\alpha_k/\alpha_j \to \infty$.

Proof. Clearly,

$$C_{P,\alpha} \;\geq\; \sup_{u \in H^1(D)} \frac{\inf_{c\in\mathbb{R}} \|u - c\|^2_{L^2(D),\alpha}}{|u|^2_{H^1(D),\alpha}}\,. \tag{2.10}$$

If α is not quasi-monotone (in the sense of Definition 2.1), then there exist indices k, j with $n > k > j \geq 1$ such that $\alpha_k = \alpha_{k-1} = \ldots = \alpha_{j+1} > \alpha_j$ and such that there is no quasi-monotone path from Y_k to $Y_{\ell^*} = Y_n$. Let us assume w.l.o.g. that $j = k-1$. Otherwise we renumber the regions. Set

$$Y_L := (\overline{Y}_1 \cup \ldots \cup \overline{Y}_{k-1})^\circ \quad \text{and} \quad Y_H := (\overline{Y}_{k+1} \cup \ldots \cup \overline{Y}_n)^\circ.$$

Then Y_k and Y_H are disconnected. Now choose $u^* \in H^1(D)$ such that

$$u^*_{|Y_H} \;=\; +1, \qquad u^*_{|Y_k} \;=\; -1, \quad \text{and} \qquad |u^*|^2_{H^1(Y_L)} \;\leq\; \beta. \tag{2.11}$$

The existence of such a function follows from the inverse trace theorem which yields $|u^*|^2_{H^1(Y_L)} \leq C_{\mathrm{tr}}\,\|u^*\|_{H^{1/2}(\partial Y_L \cap (\partial Y_H \cup \partial Y_k))} =: \beta$. The trace of u^* is constant on ∂Y_H and on ∂Y_k. Hence, the constant β depends only on the region Y_L.

Now firstly note that

$$\begin{aligned}\inf_{c\in\mathbb{R}} \|u^* - c\|^2_{L^2(D),\alpha} \;&\geq\; \inf_{c\in\mathbb{R}} \Big\{|1-c|^2\,\alpha_{k+1}\,\mathrm{meas}_d(Y_H) + |1+c|^2\,\alpha_k\,\mathrm{meas}_d(Y_k)\Big\}\\ &\geq\; \inf_{c\in\mathbb{R}} \big\{|1-c|^2 + |1+c|^2\big\}\alpha_k\,\gamma \;=\; 2\,\gamma\,\alpha_k\,.\end{aligned} \tag{2.12}$$

where $\gamma := \min(\mathrm{meas}_d(Y_H), \mathrm{meas}_d(Y_k))$. Secondly, to estimate the weighted H^1-norm of u^* from above, note first that the gradient of u^* vanishes on Y_k and on Y_H. And so using (2.11) we can conclude that

$$|u^*|^2_{H^1(D),\alpha} \;=\; |u^*|^2_{H^1(Y_L),\alpha} \;\leq\; \alpha_{k-1}\,|u^*|^2_{H^1(Y_L)} \;\leq\; \beta\,\alpha_{k-1}\,.$$

which together with (2.10) and (2.12) implies the result with $C = 2\gamma/\beta$. □

3. Weighted Poincaré inequalities for FE functions

In many applications, e.g. in the analysis of multilevel iterative methods for (1.1), it is sufficient to have Poincaré type inequalities for finite element (FE) functions. We will show now that it is possible to extend the class of weight functions α for which we can obtain weighted Poincaré inequalities to include piecewise constant functions α that clearly fall outside the original definition of quasi-monotonicity in [8] and that of the previous section.

Hence, for this section let D be a Lipschitz polygonal (polyhedral) domain in $\mathbb{R}^2$ ($\mathbb{R}^3$) and let $\{\mathcal{T}_h(D)\}_{h\in\Theta}$ be a family of shape-regular simplicial triangulations, i.e. there exists a uniform constant $c_{\mathrm{reg}} > 0$ such that for all $h \in \Theta$ and for all $\tau \in \mathcal{T}_h(D)$,

$$\frac{\mathrm{diam}(\tau)}{\rho(\tau)} \;\leq\; c_{\mathrm{reg}}\,, \tag{3.1}$$

where $\rho(\tau)$ is the diameter of the largest inscribed ball (cf. Ciarlet [6]). For each $h \in \Theta$, we define the usual space of continuous, piecewise linear finite elements

$$V_h(D) := \big\{ v \in \mathcal{C}(\overline{D}) : v_{|\tau} \text{ affine linear} \quad \forall \tau \in \mathcal{T}_h(D) \big\}.$$

Let $\alpha \in L^\infty_+(D)$ be piecewise constant again with respect to a non-overlapping partitioning of D into open, connected Lipschitz polygons (polyhedra) $\mathcal{Y} := \{Y_\ell : \ell = 1, \ldots, n\}$ such that

$$\overline{D} = \bigcup_{\ell=1}^{n} \overline{Y}_\ell \qquad \text{and} \qquad \alpha|_{Y_\ell} \equiv \alpha_\ell \tag{3.2}$$

for some constants α_ℓ. In addition we assume here that α is piecewise constant with respect to $\mathcal{T}_h(D)$, so that $\mathcal{T}_h(D)$ is aligned with $\mathcal{Y}$.

The following lemma is the crucial tool to extend our results to more general coefficients in the case of FE functions. It requires in addition that restricted to a subregion Y_ℓ the family $\{\mathcal{T}_h(D)\}_{h\in\Theta}$ is quasi-uniform, i.e. there exists a uniform constant $c_{\mathrm{quasi}} > 0$ such that for all $h \in \Theta$ and for all $\tau, \tau' \in \mathcal{T}_h(Y_\ell)$

$$\frac{\mathrm{diam}(\tau)}{\mathrm{diam}(\tau')} \;\leq\; c_{\mathrm{quasi}}\,. \tag{3.3}$$

Lemma 3.1. *Let Y be a d-dimensional simplex (triangle or tetrahedron) and let $\{\mathcal{T}_h(Y)\}_{h\in\Theta}$ be a quasi-uniform family of simplicial triangulations. Suppose $x^* \in \overline{Y}$ is an arbitrary point, and if $d = 3$, let E be an edge of tetrahedron Y. Then there exists a constant C independent of $H = \mathrm{diam}(Y)$ and h such that for all $h \in \Theta$ and for all $u \in V_h(Y)$,*

$$\|u - u(x^*)\|^2_{L^2(Y)} \;\leq\; \begin{cases} C\left(1 + \log\left(\dfrac{H}{h}\right)\right) H^2\, |u|^2_{H^1(Y)} & \text{if } d = 2, \\[2mm] C\, \dfrac{H}{h}\, H^2\, |u|^2_{H^1(Y)} & \text{if } d = 3, \end{cases}$$

and

$$\|u - \overline{u}^E\|^2_{L^2(Y)} \;\leq\; C\left(1 + \log\left(\frac{H}{h}\right)\right) H^2\, |u|^2_{H^1(Y)} \quad \text{if } d = 3.$$

Proof. The first two inequalities follow from L^∞-estimates in [34, Lemma 4.15 and inequality (4.16)]. The third inequality is proved in [34, Lemma 4.16]. For an earlier reference see [2]. The constant C depends on the ratio $\mathrm{diam}(Y)/\rho(Y)$ and on the constants c_{reg} and c_{quasi} in (3.1) and (3.3). □

Note, that clearly the dependence of the Poincaré constant on H/h gets weaker as the dimension of the manifold over which we "average" the function increases. It is linear if the dimension of the manifold is $d-3$, logarithmic if the dimension is $d-2$, and it does not depend on H/h at all if the dimension is $d-1$. The last case follows from the discussion in the previous section.

Definition 3.2. Suppose $\alpha \in L^\infty_+(D)$ satisfies (3.2), $\ell^* := \mathrm{argmax}\{\alpha_\ell\}_{\ell=1}^n$ and m is an integer between 0 and $d-1$.

(a) We call the region $P_{\ell_1,\ell_s} := (\overline{Y}_{\ell_1} \cup \overline{Y}_{\ell_2} \cup \cdots \cup \overline{Y}_{\ell_s})^\circ$, $1 \leq \ell_1, \ldots, \ell_s \leq n$, a *type-$m$ quasi-monotone path* from Y_{ℓ_1} to Y_{ℓ_s} (with respect to α), if the following two conditions hold:
   (i) for each $i = 1, \ldots, s-1$, the regions $\overline{Y}_{\ell_i}$ and $\overline{Y}_{\ell_{i+1}}$ share a common m-dimensional manifold X_i,
   (ii) $\alpha_{\ell_1} \leq \alpha_{\ell_2} \leq \cdots \leq \alpha_{\ell_s}$.

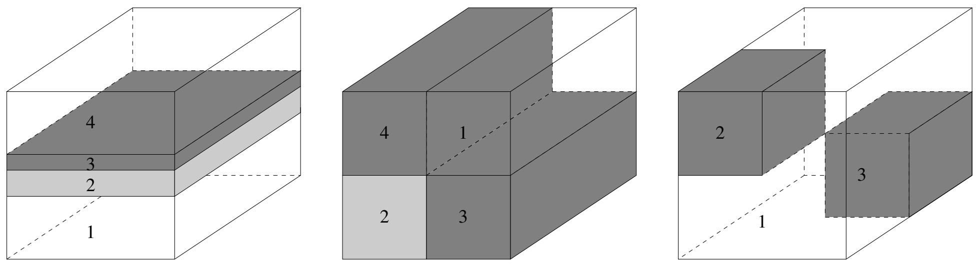


FIGURE 3. Examples of type-m quasi-monotone weight functions for $d = 3$ with $m \le 2$ in (a), with $m \le 1$ in (b) and with $m = 0$ in (c).

(b) We say that α is *type-m quasi-monotone on D*, if for all $k = 1, \dots, n$ there exists a quasi-monotone path P_{k,ℓ^*} from Y_k to Y_{ℓ^*}.

(c) Let $X^* \subset \overline{Y}_{\ell^*}$ be an m–dimensional manifold, and for each $k = 1, \dots, n$, let $c_k^{X^*} > 0$ be the best constant such that for all $h \in \Theta$

$$\|u - \overline{u}^{X^*}\|^2_{L^2(Y_k)} \;\le\; c_k^{X^*}\, \sigma^{d-m}\big(\tfrac{H}{h}\big)\, H^2\, |u|^2_{H^1(P_{k,\ell^*})} \quad \forall u \in V_h(P_{k,\ell^*}), \tag{3.4}$$

where

$$\sigma^j(x) := \begin{cases} 1 & \text{if } j = 1,\\ 1 + \log(x) & \text{if } j = 2,\\ x & \text{if } j = 3. \end{cases} \tag{3.5}$$

As before we set $C^*_{P,\alpha} := \sum_{k=1}^n c_k^{X^*}$.

Clearly a type-m quasi-monotone coefficient α is also type-$(m-1)$ quasi-monotone. In Figure 3 we see some examples. The examples in Figure 3(b–c) are clearly not quasi-monotone in the classical sense (cf. [8]), yet a discrete version of the weighted Poincaré inequality in Theorem 2.2 can be established even for these coefficients with a constant that does not depend on α.

Theorem 3.3 (discrete weighted Poincaré inequality). *Let $0 \le m \le d-1$ and let $\{\mathcal{T}_h(D)\}_{h\in\Theta}$ be quasi-uniform. If $\alpha \in L^\infty_+(D)$ is type-m quasi-monotone on D in the sense of Definition 3.2, then*

$$\inf_{c\in\mathbb{R}} \|u - c\|^2_{L^2(D),\alpha} \;\le\; C^*_{P,\alpha}\, \sigma^{d-m}\big(\tfrac{H}{h}\big)\, H^2\, |u|^2_{H^1(D),\alpha} \quad \forall u \in V_h(D). \tag{3.6}$$

where $C^*_{P,\alpha}$ and $\sigma^{d-m}(H/h)$ are defined in Definition 3.2(c).

Proof. Identical to the proof of Theorem 2.2 using (3.4) instead of (2.2). □

Let us finish this section by analysing again how the inequalities (3.4) are related to inequalities on the individual subregions Y_k.

Definition 3.4. For any bounded Lipschitz domain $Y \subset D$ resolved by $\mathcal{T}_h(D)$, and for any m–dimensional manifold $X \subset \overline{Y}$, let $C_P(Y;X) > 0$ denote the best constant such that for all $h \in \Theta$ and for all $u \in V_h(Y)$:

$$\|u - \overline{u}^X\|^2_{L^2(Y)} \;\le\; C_P(Y;X)\, \sigma^{d-m}\big(\tfrac{\mathrm{diam}(Y)}{h}\big)\, \mathrm{diam}(Y)^2\, |u|^2_{H^1(Y)}\,. \tag{3.7}$$

Lemma 3.5. *Suppose $\alpha \in L^\infty_+(D)$ is type-m quasi-monotone and P_{k,ℓ^*} is any of the paths in Definition 3.2(b) with $\ell_1 = k$ and $\ell_s = \ell^*$. For convenience let $X_0 := X_1$ and $X_s := X^*$. Then the constant $c_k^{X^*}$ in Definition 3.2(c) can be bounded by*

$$c_k^{X^*} \;\le\; 4 \sum_{i=1}^s \frac{\mathrm{meas}(Y_k)}{\mathrm{meas}(Y_{\ell_i})}\, \frac{\mathrm{diam}(Y_{\ell_i})^2}{H^2}\, \max\Big\{C_P(Y_{\ell_i}; X_{i-1}), C_P(Y_{\ell_i}; X_i)\Big\}\,.$$

Proof. The proof follows as for Lemma 2.4 using in addition that $\sigma^j(x)$ is a monotonically non-decreasing function. □

Clearly the constants $C_P(Y_{\ell_i}; X_i)$ in Lemma 3.5 (and thus $C^*_{P,\alpha}$ in Theorem 3.3) are independent of $\{\alpha_k\}_{k=1}^n$. However, to bound them independently of $\mathcal{Y}$ (i.e., geometric parameters), it is necessary to require a certain regularity of the subregions Y_k. This is technical and will be discussed in detail in Section 4.

4. Explicit dependence on geometrical parameters

Before going into the technical details, let us suppose that the partition $\mathcal{Y} = \{Y_\ell\}_{\ell=1}^n$ consists of a few well-shaped subregions and that all the interfaces X_i between adjacent subregions in Definitions 2.1 and 3.2 are well-shaped and sufficiently large. Then it follows from classical results that the constants $C_P(Y_{\ell_i}; X_i)$ and $C_P(Y_{\ell_i}, X_{i-1})$ in Definitions 2.3 and 3.4 are benign (in particular, they are independent of $\mathcal{Y}$ and h). Thanks to Lemma 2.4 this implies that the constants $C^*_{P,\alpha}$ in the weighted Poincaré inequalities in Theorems 2.2 and 3.3 are also benign.

If the assumptions above do not hold, then

(i) the number n of subregions may be large,
(ii) the shapes of the subregions Y_ℓ may be complicated, in particular long or thin and/or
(iii) the interfaces may be small compared to adjacent subregions.

In Section 4.1 below, we allow the number n to become large, but we restrict ourselves to shape-regular simplicial partitions $\mathcal{Y}$ (such that the situations in (ii) and (iii) are ruled out). We can then give explicit bounds for $C^*_{P,\alpha}$ in terms of n and $H/\eta_{\min}$, where

$$\eta_{\min} := \min_{\ell=1}^n \operatorname{diam}(Y_\ell),$$

which is a measure of the “small scale” that the coefficient introduces. In Section 4.2 we generalise the results to type-m quasi-monotone coefficients. In principal this fully describes the dependence of $C^*_{P,\alpha}$ on α, since the situations in (ii) and (iii) can always be overcome by further subdividing some regions until the partition $\mathcal{Y}$ is shape-regular. However, this can lead to pessimistic bounds. Therefore, in Sections 4.3–4.5 we show enhanced bounds for a few distinguished cases including anisotropic subregions, subregions with holes, as well as a checkerboard distribution.

For the remainder let us restrict to $d = 2$ or 3 and to piecewise constant weight functions α satisfying (2.1). To simplify the presentation we write $a \lesssim b$, if a/b can be bounded uniformly by a constant C that is independent of any parameters, in particular independent of α, $\mathcal{Y}$, H and h. Furthermore, we write $a \eqsim b$, if $a \lesssim b$ and $b \lesssim a$.

4.1. Inequalities for shape-regular partitions.

Let $\mathcal{Y} = \{Y_\ell\}_{\ell=1}^n$ be a conforming simplicial triangulation of D and define

$$\eta_\ell := \operatorname{diam}(Y_\ell), \qquad \eta := \max_{\ell=1}^n \eta_\ell, \qquad \eta_{\min} := \min_{\ell=1}^n \eta_\ell\,, \tag{4.1}$$

as well as the *shape-regularity constant*

$$c^{\mathcal{Y}}_{\text{reg}} := \max_{\ell=1}^n \frac{\operatorname{diam}(Y_\ell)}{\rho(Y_\ell)}\,. \tag{4.2}$$

Recall that a family $\{\mathcal{Y}_\eta\}_{\eta\in\Xi}$ of simplicial partitions is called *shape-regular*, if there is a uniform bound for $c^{\mathcal{Y}_\eta}_{\text{reg}}$. It is called *quasi-uniform*, if it is shape-regular and the ratios $\eta/\eta_{\min}$ are uniformly bounded. With a slight abuse of notation we will call a partition shape-regular or quasi-uniform, if it is an element of a family of such partitions.

The next lemma bounds the weighted Poincaré constant explicitly in terms of a few geometric parameters. Recall that for any quasi-monotone $\alpha \in L^\infty_+(D)$ with underlying partitioning

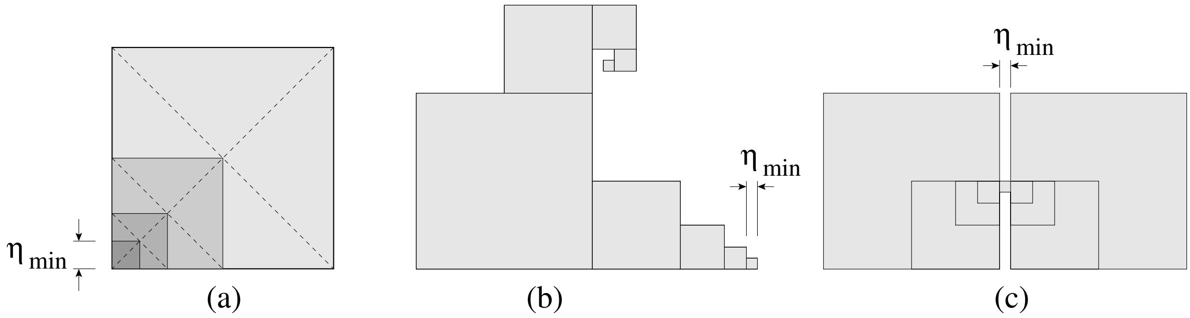


FIGURE 4. Some (more complicated) two dimensional examples with shape-regular partitions. In each case a corresponding family of partitions is defined by continuing the fractal structure and therefore halving $\eta_{\min}$. In Case (a) different colours mean different subregions and the dashed lines indicate how to further subdivide, in order to obtain a simplicial partition.

$\{Y_\ell\}_{\ell=1}^n$ and $\ell^* = \text{argmax}\{\alpha_\ell\}_{\ell=1}^n$, the length of the quasi-monotone path P_{k,ℓ^*} from Y_k to Y_{ℓ^*} in Definition 2.1 is denoted by s_k.

Lemma 4.1. *Let $\mathcal{Y} = \{Y_\ell\}_{\ell=1}^n$ be a shape-regular simplicial partition of D and let $\alpha \in L_+^\infty(D)$ be quasi-monotone with respect to $\mathcal{Y}$ (in the sense of Definition 2.1 with X^* a facet of the simplex Y_{ℓ^*}). Then*

$$C_{P,\alpha}^* \;\leq\; 2^{d+1}\,(c_{\text{reg}}^{\mathcal{Y}})^{d-1} \sum_{k=1}^n \frac{s_k\,\text{meas}_d(Y_k)}{H^2\,\eta_{\min}^{d-2}}\,.$$

Proof. The proof is based on Lemma 2.4 and we adopt the same notation. We fix $k \in \{1,\ldots,n\}$ and choose a quasi-monotone path $P_{k,\ell^*} = (\overline{Y}_{\ell_1} \cup \ldots \cup \overline{Y}_{\ell_{s_k}})^\circ$ of length s_k. It follows from Lemma A.1 in the Appendix, that $\max\{C_P(Y_{\ell_i}; X_{i-1}), C_P(Y_{\ell_i}; X_i)\} \leq 1$. Due to Lemma A.2 in the Appendix,

$$\frac{\text{diam}(Y_\ell)^2}{\text{meas}_d(Y_\ell)} \;\leq\; 2^{d-1}\,(c_{\text{reg}}^{\mathcal{Y}})^{d-1}\,\eta_\ell^{2-d}\,.$$

Thus, Lemma 2.4 implies that

$$c_k^{X^*} \;\leq\; 4\sum_{i=1}^{s_k} 2^{d-1}\,(c_{\text{reg}}^{\mathcal{Y}})^{d-1}\,\frac{\text{meas}_d(Y_k)}{H^2}\,\eta_{\ell_i}^{2-d}. \tag{4.3}$$

Since $d \geq 2$ the result follows from the definition of $C_{P,\alpha}^*$ in Definition 2.1. □

The following corollary gives the worst case scenario.

Corollary 4.2. *With the assumptions of Lemma 4.1*

$$C_{P,\alpha}^* \;\lesssim\; (H/\eta_{\min})^{2(d-1)}\,.$$

If we assume in addition that $s_k \lesssim H/\eta_{\min}$, for all $k = 1,\ldots,n$, i.e. none of the quasi-monotone paths follows a plane (space) filling curve, then

$$C_{P,\alpha}^* \;\lesssim\; (H/\eta_{\min})^{d-1}\,.$$

Proof. Note that $\sum_{k=1}^n \text{meas}_d(Y_k) = \text{meas}_d(D) \leq H^d$. Due to shape regularity $s_k \leq n \lesssim (H/\eta_{\min})^d$ (at most). Hence, the result follows from Lemma 4.1. □

Obviously, the results above extend straightforwardly to the case of polygonal (polyhedral) partitions $\mathcal{Y}$, where each subregion Y_ℓ consists of a small number of simplices, such that the resulting simplicial partition of D is shape regular and conforming. In the examples below we will often make use of this fact.

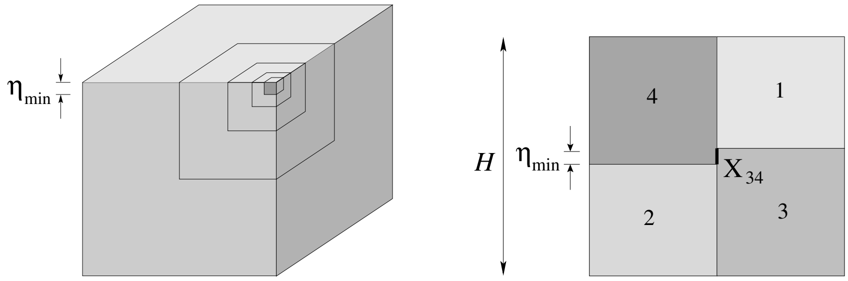


FIGURE 5. *Left:* Example with shape-regular polyhedral partition, consisting of a small cube and nested Fichiera corners. *Right:* Coefficient distribution with staggered structure. (The largest coefficient is in region Y_4.)

Example 4.3. Let $d = 2$ and consider the three domains shown in Figure 4. Note that in all three cases the assumptions of Lemma 4.1 are fulfilled, the underlying simplicial partition (only shown for (a)) is shape-regular, $\mathrm{meas}_2(D) \approx H^2$, and $\eta_{\min} \approx 2^{-n}H$. Since $\max_{k=1}^n s_k \le n \lesssim \log_2(H/\eta_{\min})$ in each of these cases, it follows from Lemma 4.1 that

$$C^*_{P,\alpha} \;\lesssim\; 1 + \log\Big(\frac{H}{\eta_{\min}}\Big).$$

Remark 4.4. Example 4.3 shows that the (standard) Poincaré constant $C_P(D)$ of the two-dimensional "dumbbell" domain in Figure 4(c) is $\mathcal{O}(1+\log(H/\eta_{\min}))$. Note that the isoperimetric constant (often used to bound $C_P(D)$, cf. [7, 19, 20]) is $\mathcal{O}(H/\eta_{\min})$ and thus yields a pessimistic bound for $C_P(D)$.

Example 4.5. Let now $d = 3$ and consider the domain in Figure 5 (left) with Y_1 being the small cube in the top corner and the remaining subregions numbered away from Y_1, such that $\eta_k \approx 2^k \, \eta_{\min}$.

Let us first consider the case that $\ell^* = 1$, i.e. the largest coefficient is in the small cube. Let k be fixed, then $s_k = k$ and $\ell_i = k+1-i$. It follows from inequality (4.3) in the proof of Lemma 4.1 that

$$c_k^{X^*} \;\lesssim\; \sum_{i=1}^{s_k} \frac{\eta_k^3}{H^2}\,\eta_{k+1-i}^{-1} \;\lesssim\; \sum_{i=1}^{s_k} \frac{4^k\,\eta_{\min}^2}{H^2} \sum_{i=1}^{k} 2^i \;\lesssim\; \frac{\eta_{\min}^2}{H^2}\,8^k.$$

Since $n \approx \log_2(H/\eta_{\min})$, we get $8^n \approx (H/\eta_{\min})^3$ and thus

$$C^*_{P,\alpha} \;\lesssim\; \frac{\eta_{\min}^2}{H^2} \sum_{k=1}^{n} 8^k \;\lesssim\; \frac{H}{\eta_{\min}}\,.$$

If, on the other hand, the largest coefficient value is attained in the largest domain, i.e. $\ell^* = n$, then for fixed k, we have $s_k = n - k + 1$ and $\ell_i = k - 1 + i$. And so, using again inequality 4.3 in the proof of Lemma 4.1, we get

$$c_k^{X^*} \;\lesssim\; \sum_{i=1}^{n-k+1} \frac{\eta_k^3}{H^2}\,\eta_{k-1+i}^{-1} \;\lesssim\; \frac{\eta_{\min}^2}{H^2}\,4^k \sum_{i=1}^{n-k+1} 2^{1-i} \;\lesssim\; \frac{\eta_{\min}^2}{H^2}\,4^k \;\lesssim\; 4^{k-n}\,,$$

where in the last step we used that $\eta_{\min} \approx 2^{-n}H$. Hence, for any n,

$$C^*_{P,\alpha} \;\lesssim\; 1.$$

In the same way, we can also show that $C_{P,\alpha} \lesssim 1$ for the domains in Figure 4(a) and Figure 4(b), if the largest coefficient is attained in the largest subregion.

Note that the examples in this section are not artificial. They arise naturally when interfaces between perfectly well-shaped coefficient regions are *small* compared to the size of the regions, see e.g. Figure 5 (right). This case can often be treated by *artificially* subdividing some subregions further in a suitable way.

Example 4.6. Consider the scenario in Figure 5 (right). The quasi-monotone path $P_{3,4}$ from Y_3 to Y_4 contains the interface $X_{3,4}$ which has $\mathrm{diam}(X_{3,4}) = \eta_{\min} \ll H$. However, subdividing both Y_3 and Y_4 further as shown in Figure 4(a) we get as in Example 4.3

$$C^*_{P,\alpha} \lesssim 1 + \log\Big(\frac{H}{\eta_{\min}}\Big).$$

4.2. **Inequalities for FE functions on shape-regular partitions.** In this subsection, we generalise the explicit results of the previous section to the discrete case and discuss a few particularities.

It was important in Section 4.1 that the $(d-1)$-dimensional manifold X^* was chosen to be a $(d-1)$–dimensional facet of the simplex Y_{ℓ^*}, i.e. an edge in 2D or a face in 3D. In this section, for type-m quasi-monotone coefficients, we choose X^* to be an *m-facet* of the simplex Y_{ℓ^*}.

Definition 4.7. Let $\mathcal{Y} = \{Y_\ell\}_{\ell=1}^n$ be a simplicial partition of D. Then each boundary ∂Y_ℓ is the union of

- 0-*facets*: the vertices of the simplex,
- 1-*facets*: the edges of the simplex,
- 2-*facets*: the faces of the simplex (if $d = 3$).

It is straightforward to extend the results from Section 4.1 to type-m quasi-monotone coefficients, provided the mesh $\mathcal{T}_h(D)$ resolves the partition $\mathcal{Y}$ and is quasi-uniform on each of the simplices Y_ℓ. Doing this carefully we even get an enhanced bound compared to Theorem 3.3. Let $h_\ell := \max_{\tau\subset Y_\ell} \mathrm{diam}(\tau)$ be the local mesh size on Y_ℓ and recall that s_k is the length of the type-m quasi-monotone path P_{k,ℓ^*} defined in Definition 3.2.

Lemma 4.8. *For $d > 1$, let $\mathcal{Y} = \{Y_\ell\}_{\ell=1}^n$ be a shape-regular simplicial partition of D and let $\mathcal{T}_h(D)$ be such that its restriction $\mathcal{T}_h(Y_\ell)$ is quasi-uniform for all $\ell = 1, \ldots, n$. If $\alpha \in L^\infty_+(D)$ is type-m quasi-monotone with respect to $\mathcal{Y}$ (in the sense of Definition 3.2) and if X^* is an m-facet of the simplex Y_{ℓ^*}), then*

$$\inf_{c\in\mathbb{R}} \|u - c\|^2_{L^2(D),\alpha} \leq C^{*,m}_{P,\alpha}\, H^2\, |u|^2_{H^1(D),\alpha} \qquad \forall u \in V^h(D), \tag{4.4}$$

where $C^{*,m}_{P,\alpha} \lesssim \sigma^{d-m}\Big(\max_{\ell=1}^n \frac{\eta_\ell}{h_\ell}\Big) \sum_{k=1}^n s_k \, \frac{\mathrm{meas}_d(Y_k)}{H^2\,\eta_{\min}^{d-2}}$. *The hidden constant depends on $c^{\mathcal{Y}}_{\mathrm{reg}}$ and on the constant in Lemma 3.1.*

Proof. The proof follows the same lines as that of Lemma 4.1. Let $c = \overline{u}^{X^*}$. Since $c^{\mathcal{Y}}_{\mathrm{reg}} \eqsim 1$ it follows from Lemma 3.1 that the discrete Poincaré constants $C_P(Y_{\ell_i}; X_{i-1})$ and $C_P(Y_{\ell_i}; X_i)$ are both $\mathcal{O}\big(\sigma^{d-m}(\eta_{\ell_i}/h_{\ell_i})\big)$, where σ^{d-m} is as defined in (3.5). The result then follows as in the proof of Lemma 4.1. □

As in Section 4.1, if we exclude pathological examples with type-m quasi-monotone paths P_{k,ℓ^*} that follow plane (space) filling curves, Lemma 4.8 yields a worst case scenario of

$$C^{*,m}_{P,\alpha} \lesssim \Big(\frac{H}{\eta_{\min}}\Big)^{d-1} \sigma^{d-m}\Big(\max_{\ell=1}^n \frac{\eta_\ell}{h_\ell}\Big).$$

To apply Lemma 4.1 it was crucial that each Y_ℓ in the partition was a simplex. As mentioned several times, a polygonal (polyhedral) region Y_ℓ that is not simplicial can always be artificially subdivided into a set of simplicial ones. However, it is often difficult to guarantee that a mesh $\mathcal{T}_h(D)$ that is aligned with the original partition is also aligned with the artificial simplicial subpartition, and we would not want to impose such a condition. The next lemma shows that for any polygonal (polyhedral) region Y that is the union of a small number of simplices, it suffices that there exists a quasi-uniform triangulation $\widetilde{\mathcal{T}}_h(Y)$ that is aligned with the simplicial subpartition of Y and has the same mesh size as $\mathcal{T}_h(Y)$, such that the results of Lemma 4.1 hold.

Lemma 4.9. *Let Y be the union of a small number of shape-regular and quasi-uniform simplices $T_1, \ldots, T_p$ and set $H := \operatorname{diam}(Y)$. Let $\mathcal{T}_h(Y)$ be a quasi-uniform simplicial triangulation of Y (not necessarily aligned with $\{T_i\}_{i=1}^p$) and let $X \subset \partial Y$ be an m-facet of one of the simplices T_i (note that X is resolved by $\mathcal{T}_h(Y)$). Then*

$$\|u - \overline{u}^X\|_{L^2(Y)}^2 \;\lesssim\; \sigma^{d-m}\big(\tfrac{H}{h}\big)\, H^2\, |u|_{H^1(Y)}^2 \quad \forall u \in V_h(Y).$$

The hidden constant depends on c, on the number of simplices p, on the constant C in Lemma 3.1, and on the shape-regularity constants of $\mathcal{T}_h(Y)$ and $\{T_i\}_{i=1}^p$.

Proof. It is always possible to refine the simplices $T_1, \ldots, T_p$ to obtain a quasi-uniform simplicial triangulation $\widetilde{\mathcal{T}}_h(Y)$ with mesh size h that coincides with $\mathcal{T}_h(Y)$ on the boundary ∂Y and that has a shape-regularity constant which is bounded by the shape-regularity constants of $\mathcal{T}_h(Y)$ and $\{T_i\}_{i=1}^p$. Let $\widetilde{V}_h(Y)$ be the corresponding FE space of continuous piecewise linear functions. Since $\widetilde{\mathcal{T}}_h(Y)$ is aligned with $\{T_i\}_{i=1}^p$ we can apply Lemma 4.8 (with $\alpha \equiv 1$) to get

$$\|u - \overline{u}^X\|_{L^2(Y)}^2 \;\lesssim\; \sigma^{d-m}\big(\tfrac{H}{h}\big)\, H^2\, |u|_{H^1(Y)}^2 \quad \forall u \in \widetilde{V}_h(Y). \tag{4.5}$$

To show that an equivalent statement holds for functions $u \in V_h(Y)$ we make use of the Scott-Zhang operator from [33] (see also [3]). Let $V_h(\partial Y)$ be the trace space of $V_h(Y)$, which is identical to the trace space of $\widetilde{V}_h(Y)$. There exists an operator $\Pi_h : H^1(Y) \to \widetilde{V}_h(Y)$ such that for all $v \in H^1(Y)$ with $v_{|\partial Y} \in V_h(\partial Y)$

$$(\Pi_h v)_{|\partial Y} \;=\; v_{|\partial Y}\,, \tag{4.6}$$

$$\|v - \Pi_h v\|_{L^2(Y)} \;\le\; C_{\rm sc}\, h\, |v|_{H^1(Y)}\,, \tag{4.7}$$

$$|\Pi_h v|_{H^1(Y)} \;\le\; C_{\rm sc}\, |v|_{H^1(Y)}\,. \tag{4.8}$$

The operator is constructed by local averages over $(d-1)$-dimensional manifolds and the constant $C_{\rm sc}$ only depends on the shape-regularity constant of $\widetilde{\mathcal{T}}_h(Y)$.

Let $u \in V_h(Y)$ be arbitrary but fixed. Then, due to (4.6), $\overline{\Pi_h u}^X = \overline{u}^X$ and it follows from (4.5) and (4.7) that

$$\begin{aligned}\|u - \overline{u}^X\|_{L^2(Y)} \;&\le\; \|u - \Pi_h u\|_{L^2(Y)} + \|\Pi_h u - \overline{\Pi_h u}^X\|_{L^2(Y)}\\ &\lesssim\; h|u|_{H^1(Y)} + \sqrt{\sigma^{d-m}\big(\tfrac{H}{h}\big)}\, H\, |\Pi_h u|_{H^1(Y)}\,.\end{aligned}$$

Clearly, $h \le H$ and $\sigma^{d-m}(\frac{H}{h}) \ge 1$, and so the result follows from (4.8). ☐

4.3. **Anisotropic subregions.** In this subsection we treat cases where the partition $\mathcal{Y}$ contains anisotropic subregions. We will see that it is often advantageous *not* to further subdivide this into a shape regular partition. We start by showing an elementary result for the Poincaré constant of a parallelepiped.

Lemma 4.10. *Let $\{\vec{e}_i\}_{i=1}^d$ be a (normalised) basis of $\mathbb{R}^d$ and let Y be the parallelogram/parallelepiped $\big\{\sum_{i=1}^d \beta_i\, \vec{e}_i : \beta_i \in (0, L_i)\big\}$. If X is one of the facets (edges/faces) of Y, then*

$$C_P(Y;X) \;\approx\; 1,$$

and the hidden constant is independent of the aspect ratios L_i/L_j and of the angles between $\vec{e}_i$ and $\vec{e}_j$, for any $1 \le i, j \le d$.

Proof. The result can easily be shown by transforming Y to the (isotropic) reference cube $Q = (0,1)^d$ using the *linear* transformation $F(x) = J^{-1}\, x$ where $J = (L_1\vec{e}_1|\cdots|L_d\vec{e}_d)$. ☐

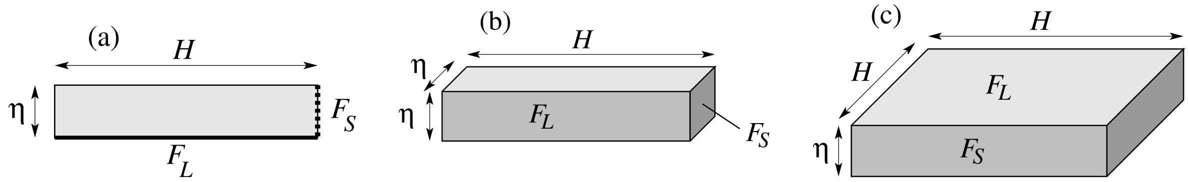


FIGURE 6. Three model cases of anisotropic domains in two (Case (a)) and three dimensions (Cases (b) and (c)).

Example 4.11. For any of the regions Y in Figure 6(a–c) and for any $(d-1)$–facet X of Y, Lemma 4.10 implies

$$C(Y, X) \eqsim 1,$$

independent of the aspect ratio H/η.

Example 4.12. Let Y be one of the two "annular" subregions shown in Figure 7 (left/middle), and let X be an edge of length H (left figure) or a face of area H^2 (middle figure). Then $C_P(Y;X) \eqsim 1$. This can be shown by further subdividing the subregions into a few anisotropic rectangles/cuboids and using Lemma 2.4 (with $D = Y$ and $X^* = X$) together with the estimates in Example 4.11. Such estimates can already be found in [24].

Our next example will be Figure 7 (right), where a piecewise constant coefficient increases gradually towards an edge of a cube in 3D. To get an optimal bound in this case is surprisingly difficult. We require a variation of Lemma 2.4.

Lemma 4.13. *Let $\alpha \in L^\infty_+(D)$ be quasi-monotone with respect to a partition $\mathcal{Y}$. Let ℓ^* be the index of the region where the maximum is attained and let X^* be a $(d-1)$-dimensional manifold in ∂Y_{ℓ^*}. For each $k = 1, \ldots, n$, let X_k be a $(d-1)$-dimensional manifold in ∂Y_k and let P_{k,ℓ^*} be the quasi-monotone path from Definition 2.1. Then,*

$$C^*_{P,\alpha} \lesssim \max_{k=1}^{n} \Big\{ \frac{\mathrm{diam}(Y_k)^2}{H^2} C_P(Y_k; X_k) \Big\} + \sum_{k=1}^{n} \frac{\mathrm{meas}_d(Y_k)}{\mathrm{meas}_d(P_{k,\ell^*})} \frac{\mathrm{diam}(P_{k,\ell^*})^2}{H^2} \Big\{ C_P(P_{k,\ell^*}; X_k) + C_P(P_{k,\ell^*}; X^*) \Big\}.$$

Proof. The proof follows that of Lemma 2.4. Let $1 \le k \le n$ be fixed. Then,

$$\tfrac{1}{2} \, \|u - \overline{u}^{X^*}\|^2_{L^2(Y_k),\alpha} \;\le\; \alpha_k \, \|u - \overline{u}^{X_k}\|^2_{L^2(Y_k)} + \alpha_k \, \mathrm{meas}_d(Y_k) \, |\overline{u}^{X_k} - \overline{u}^{X^*}|^2$$

For the first summand, we have

$$\alpha_k \, \|u - \overline{u}^{X_k}\|^2_{L^2(Y_k)} \;\le\; C_P(Y_k; X_k) \, \frac{\mathrm{diam}(Y_k)^2}{H^2} \, H^2 \, |u|^2_{H^1(Y_k),\alpha} \, .$$

The second summand can be bounded in the same way as (2.7) (but with P_{k,ℓ^*} instead of Y_{ℓ_i} and with X_k and X^* instead of X_{i-1} and X_i). To conclude the proof we have to use quasi-monotonicity and sum the two bounds over k. □

Example 4.14. For the scenario in Figure 7 (right), we have

$$C^*_{P,\alpha} \lesssim 1 + \log\Big(\frac{H}{\eta}\Big).$$

To see this, we first consider the subdivision $\{Y_\ell\}_{\ell=1}^n$ with $n \eqsim 1 + \log(H/\eta)$ depicted in Figure 7 (right) and apply Lemma 4.13 with X^* one of the long and thin faces of Y_{ℓ^*}. Clearly, $C_P(Y_k; X_k) \eqsim 1$ and $C_P(P_{k,\ell^*}; X_k) \eqsim 1$ because these regions consist of a few cuboids and X_k is one of its faces. Hence, it remains to investigate $C_P(P_{k,\ell^*}; X^*)$. First we consider the case $k = 1$, where $P_{1,\ell^*} = D$.

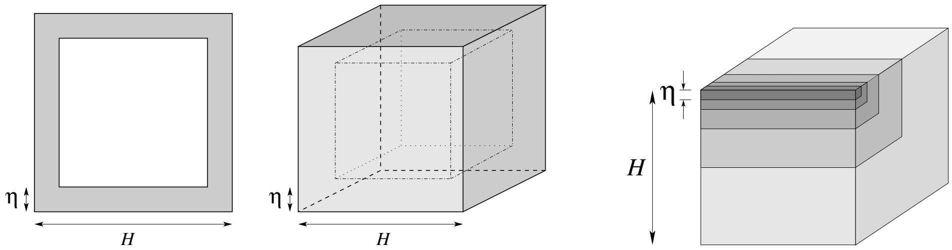


FIGURE 7. *Left/middle:* "Annular" subregions in Example 4.12 in two and three dimensions. The smaller cube sketched inside is cut out from the larger cube. *Right:* Piecewise constant coefficient distribution increasing gradually towards an edge in 3D.

In the limit case $\eta \to 0$, the face X^* collapses to an edge E of D. Here we can make use of Lemma 3.1 which can straightforwardly be generalised to cubes. Let $\mathcal{T}_h$ be an auxiliary quasi-uniform triangulation of D such that the face X^* is resolved by just one layer of element faces ($h \eqsim \eta$) and let $V_h(D)$ denote the corresponding piecewise linear finite element space. As in Lemma 4.9 we make use of a Scott-Zhang type quasi-interpolation operator (see [33, 3]), i.e. there exists an operator $\Pi_h : H^1(D) \to V_h(D)$ such that for all $v \in H^1(D)$,

$$\begin{aligned} \overline{\Pi_h v}^E &= \overline{v}^{X^*}, \\ \|v - \Pi_h v\|_{L^2(D)} &\leq C_{\rm sc}\, h\, |v|_{H^1(D)}, \\ |\Pi_h v|_{H^1(D)} &\leq C_{\rm sc}\, |v|_{H^1(D)}, \end{aligned}$$

with a uniform constant $C_{\rm sc}$. The interpolator is constructed by defining the values at the mesh nodes by averages over suitable $(d-1)$-dimensional manifolds. For the nodes in $\overline{X}^*$, we choose element faces in X^* such that $(\Pi_h v)_{|X^*}$ is constant in the direction perpendicular to E, and so $\overline{\Pi_h v}^E = \overline{v}^{X^*}$. We now obtain from the properties of Π_h and from Lemma 3.1 that for all $u \in H^1(D)$,

$$\begin{aligned} \|u - \overline{u}^{X^*}\|^2_{L^2(D)} &\lesssim \|u - \Pi_h u\|^2_{L^2(D)} + \|\Pi_h u - \overline{\Pi_h u}^E\|^2_{L^2(D)} \\ &\lesssim h^2\, |u|^2_{H^1(D)} + H^2\,(1 + \log(H/h))\, |\Pi_h u|^2_{H^1(D)} \\ &\lesssim H^2\,(1 + \log(H/h))\, |u|^2_{H^1(D)}\,. \end{aligned}$$

Hence, since $h \eqsim \eta$, we get that $C_P(P_{1,\ell^*}; X^*) \lesssim 1 + \log(H/\eta) \eqsim n$.

Next we investigate P_{k,ℓ^*}, for $k > 1$. Consider the linear transformation from the reference cube $\widehat{Q}$ to P_{k,ℓ^*}. This consists simply in multiplying two of the coordinates by $2^{-k}\, H^{-1}$ and the remaining one by H^{-1}. Then

$$\|\widehat{u}\|^2_{L^2(\widehat{Q})} = \frac{\mathrm{meas}_d(\widehat{Q})}{\mathrm{meas}_d(P_{k,\ell^*})}\, \|u\|^2_{L^2(P_{k,\ell^*})} \quad \text{and} \quad |\widehat{u}|^2_{H^1(\widehat{Q})} \leq \frac{\mathrm{meas}_d(\widehat{Q})}{\mathrm{meas}_d(P_{k,\ell^*})}\, |u|^2_{H^1(P_{k,\ell^*})}\,,$$

because the spectral norm of the Jacobian is ≤ 1. On $\widehat{Q}$ we can choose a quasi-uniform mesh with mesh size $h \eqsim 2^{-(n+1-k)}$ and apply the arguments from before (with $D = \widehat{Q}$ and $H = 1$) to obtain

$$C_P(P_{k,\ell^*}; X^*) \lesssim 1 + \log(1/h) \eqsim 1 + \log(2^{n+1-k}) \eqsim n + 1 - k.$$

Putting all the estimates together finally yields

$$C^*_{P,\alpha} \lesssim 1 + \sum_{k=1}^{n} \frac{(2^{-k}H)^2}{H^2}\big(1 + (n+1-k)\big) \eqsim \sum_{k=1}^{n} 4^{-k}(1 - n + k) \eqsim n,$$

where $n \eqsim 1 + \log(H/\eta)$.

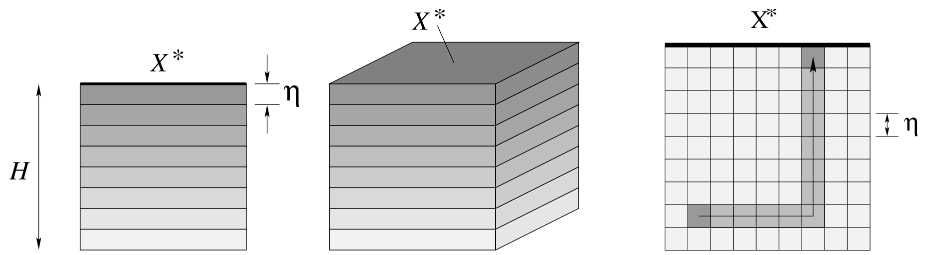


FIGURE 8. *Left/middle:* Layered coefficient distributions in two and three dimensions. *Right:* Partitioning and quasi-monotone paths for Example 4.17.

Unfortunately, using Lemma 4.13 for the layered coefficient distribution in Figure 8 (left/middle) leads to a sub-optimal bound $C^*_{P,\alpha} \lesssim 1 + \log(H/\eta)$ (that grows with the number of layers). The following alternative theory to Lemma 2.4 and Lemma 4.13 (first given in the appendix of [24]) leads to optimal bounds even in these cases.

Here, we actually do need to further partition the anisotropic subregions such that $\{Y_\ell\}_{\ell=1}^n$ is simplicial and quasi-uniform. Furthermore, X^* has to be the union of a subset $\{F_j\}_{j=1}^J$ of the $(d-1)$–facets of the simplices Y_ℓ (edges for $d = 2$ and faces for $d = 3$). For simplicity we assume that the numbering is such that Y_j is the (unique) simplex whose boundary contains F_j, for all $j = 1, \ldots, J$.

Lemma 4.15. *Let $\mathcal{Y} := \{Y_\ell\}_{\ell=1}^n$ be simplicial and quasi-uniform with mesh size $\eta > 0$, and let $\overline{X}^* = \bigcup_{j=1}^J \overline{F}_j$ such that $F_j \subset \overline{Y}_j$. For any $k \in \mathcal{I} := \{1, \ldots, n\}$ and $j \in \mathcal{J} := \{1, \ldots, J\}$, let $P_{k,j}$ be a path from Y_k to Y_j. Then*

$$\int_{F_j} \int_{Y_k} \left|u(x) - u(y)\right|^2 dy\, ds_x \;\lesssim\; s_{k,j}\, \eta^{d+1}\, |u|^2_{H^1(P_{k,j})} \qquad \forall u \in H^1(P_{k,j}),$$

where $s_{k,j}$ is the length of the path $P_{k,j}$.

Proof. Note first that

$$\begin{aligned}\int_{F_j} \int_{Y_k} \left|u(x) - u(y)\right|^2 dy\, ds_x \;&\lesssim\; \int_{F_j} \int_{Y_k} \left|u(x) - \overline{u}^{F_j}\right|^2 + \left|\overline{u}^{F_j} - u(y)\right|^2 dy\, ds_x \\ &\lesssim\; \mathrm{meas}_{d-1}(F_j)\, \|u - \overline{u}^{F_j}\|^2_{L^2(Y_k)} + \mathrm{meas}_d(Y_k)\, \|u - \overline{u}^{F_j}\|^2_{L^2(F_j)}\,. \end{aligned} \tag{4.9}$$

It follows from Lemma 4.1 (with $D = P_{k,j}$ and $X^* = F_j$) that

$$\|u - \overline{u}^{F_j}\|^2_{L^2(Y_k)} \;\lesssim\; s_{k,j}\, \frac{\mathrm{meas}_d(Y_k)}{\eta^{d-2}}\, |u|^2_{H^1(P_{k,j})}\,. \tag{4.10}$$

Also, by transformation to the reference simplex we get that

$$\|u - \overline{u}^{F_j}\|_{L^2(F_j)} \;\lesssim\; \eta\, |u|^2_{H^1(Y_j)}\,. \tag{4.11}$$

Substituting these last two bounds into (4.9), the final result follows from the fact that by assumption $\mathrm{meas}_d(Y_k) \eqsim \eta^d$ and $\mathrm{meas}_{d-1}(F_j) \eqsim \eta^{d-1}$. □

Lemma 4.16. *Under the assumptions of Lemma 4.15, let $\alpha \in L^\infty_+(D)$ be quasi-monotone with respect to $\mathcal{Y}$ (in the sense of Definition 3.2) and let each $P_{k,j}$ be quasi-monotone with respect to α. Then*

$$C^*_{P,\alpha} \;\lesssim\; \frac{s_{\max}\, r_{\max}\, \eta^{d+1}}{\mathrm{meas}_{d-1}(X^*)\, H^2}\,,$$

where $s_{\max} := \max\{s_{k,j} : (k,j) \in \mathcal{I} \times \mathcal{J}\}$ and

$$r_{\max} := \max_{i \in \mathcal{I}} \left|\left\{(k,j) \in \mathcal{I} \times \mathcal{J} : Y_i \subset P_{k,j}\right\}\right|\,,$$

i.e. the maximum number of times any of the simplices Y_i is contained in a path.

Proof. W.l.o.g. let $u \in H^1(D)$ with $\overline{u}^{X^*} = 0$ be arbitrary but fixed. We now integrate the identity $u(x)^2 - 2u(x)u(y) + u(y)^2 = (u(x) - u(y))^2$ over X^* with respect to x, multiply it by $\alpha(y)$, and finally integrate over D with respect to y:

$$\int_D \alpha(y)\,dy\,\|u\|^2_{L^2(X^*)} - 2\int_{X^*} u(x)\,ds_x \int_D \alpha(y)\,u(y)\,dy + \\ +\ \mathrm{meas}_{d-1}(X^*)\,\|u\|^2_{L^2(D),\alpha} \ = \ \int_{X^*}\int_D \alpha(y)\,\big|u(x) - u(y)\big|\,dy\,ds_x\,.$$

The middle term on the left hand side vanishes since $\overline{u}^{X^*} = 0$. Thus,

$$\begin{aligned}\mathrm{meas}_{d-1}(X^*)\,\|u\|^2_{L^2(D),\alpha} \ &\le\ \int_{X^*}\int_D \alpha(y)\,\big|u(x) - u(y)\big|\,dy\,ds_x \\ &=\ \sum_{k\in\mathcal{I}}\sum_{j\in\mathcal{J}} \alpha_k \int_{F_j}\int_{Y_k} \big|u(x) - u(y)\big|\,dy\,ds_x\,.\end{aligned}$$

Using Lemma 4.15, quasi-monotonicity and the definitions of $s_{\max}$ and $r_{\max}$

$$\begin{aligned}\mathrm{meas}_{d-1}(X^*)\,\|u\|^2_{L^2(D),\alpha} \ &\lesssim\ \sum_{k\in\mathcal{I}}\sum_{j\in\mathcal{J}} s_{k,j}\,\eta^{d+1}\,|u|^2_{H^1(P_{k,j}),\alpha} \\ &\le\ s_{\max}\,\eta^{d+1}\sum_{i\in\mathcal{I}} \big|\big\{(k,j)\in\mathcal{I}\times\mathcal{J} : Y_i \subset P_{k,j}\big\}\big|\ |u|^2_{H^1(Y_i),\alpha} \\ &\le\ s_{\max}\,r_{\max}\,\eta^{d+1}\,|u|^2_{H^1(D),\alpha}\end{aligned}$$

which concludes the proof. □

Obviously, the statements of Lemma 4.15 and Lemma 4.16 apply also for non-simplicial partitions (e.g. quadrilateral or hexahedral), if each region Y_i, $i \in \mathcal{I}$, consists of a few simplices and the resulting simplicial mesh is quasi-uniform.

Example 4.17. For the two scenarios in Figure 8 (left/middle), we have

$$C^*_{P,\alpha} \ \lesssim\ 1.$$

We only give the proof for $d = 2$. The case $d = 3$ is analogous.

We subdivide each anisotropic region in Figure 8 (left), such that the resulting partition $\mathcal{Y}$ consists of $(H/\eta)^2$ square regions Y_k, as shown in Figure 8 (right). The manifold X^* (on the top of ∂D) with $\mathrm{meas}_{d-1}(X^*) = H$ is the union of H/η edges F_j. By using generic "L"–shaped paths $P_{k,j}$ from Y_k to F_j as depicted in Figure 8 (right), for any pair $(k,j) \in \mathcal{I}\times\mathcal{J}$, it is easy to see that (i) each of the paths is quasi-monotone with respect to the given coefficient distribution in Figure 8 (left), (ii) $s_{\max} \eqsim H/\eta$ and (iii) $r_{\max} \eqsim (H/\eta)^2$. Therefore it follows from Lemma 4.16 that $C^*_{P,\alpha} \lesssim 1$.

4.4. **Subregions with inclusions.** As an example of this type we consider the region depicted in Figure 1(c) with a large number of square inclusions and choose X^* to be a boundary edge of D of length $\eqsim H$.

To bound the weighted Poincaré constant $C^*_{P,\alpha}$ for this case, we treat all the inclusions as one subregion Y_1 and the remainder as Y_2. Then, the path $P_{12} = D$ and so

$$c_1^* \ \lesssim\ C_P(D, X^*) \ \lesssim\ 1.$$

To get a bound for the Poincaré constant $c_2^* = C_P(Y_2; X^*)$ of the *perforated* domain Y_2 without the inclusions, we will use Lemma 4.16. It is straightforward to find a quasi-uniform (square) partition $\{\widetilde{Y}_i\}_{i=1}^n$ of Y_2 with mesh size equal to the diameter η of the holes (see Figure 1(c)). We construct a (quasi-monotone) path from each region $\widetilde{Y}_i$ to one of the faces $F_j \subset X^*$ by following (essentially) the same construction as in Example 4.17 (with some small modifications at the

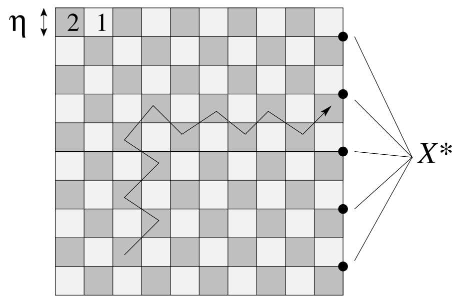


FIGURE 9. The checkerboard distribution.

start and at the end of the path). It is easy to see that again $s_{\max} \lesssim H/\eta$ and $r_{\max} \lesssim (H/\eta)^2$. Hence,

$$C_P(Y_2; X^*) \lesssim 1 \quad \text{and so} \quad C^*_{P,\alpha} \lesssim 1.$$

If there are p distinct values in the inclusions, following the same technique we see that

$$C^*_{P,\alpha} \lesssim p.$$

On first glance this would suggest, that in the worst case $C^*_{P,\alpha} \lesssim n$, but this is not quite true. Using the concept of macroscopically quasi-monotone coefficients (introduced in Section 2.2) we may combine subregions with weights of similar size, even if they are not connected. Assume, for example, that the values of α range from $\alpha_1 = 10^{-6}$ to $\alpha_n = 1$, where Y_n is now the perforated (background) region. If we combine all subregions with values in $[10^{-i}, 10^{-i+1}]$ into one subregion, we have a local variation of 10 in each subregion. Therefore, since there are 6 such combined subregions,

$$C^*_{P,\alpha} \lesssim 60$$

uniformly, even for $n \to \infty$. We note that estimates for $C^*_{P,\alpha}$ for this example have been shown in [11, Lemma 4], but they depend on the number n of inclusions and are not explicit in the geometric parameters.

4.5. **The checkerboard distribution.** Our last type of example is that of checkerboard-type distributions, as depicted in Figure 9. We will show that the discrete Poincaré inequality (4.4) for the coefficient in Figure 9 holds with

$$C^{*,m}_{P,\alpha} \lesssim 1 + \log\Big(\frac{\eta}{h}\Big).$$

In a similar way to Lemma 4.16 we can prove the following bound for $C^{*,m}_{P,\alpha}$ in (4.4) in Lemma 4.8.

Lemma 4.18. *For $d > 1$, let $\mathcal{Y} = \{Y_\ell\}_{\ell=1}^n$ be a quasi-uniform simplicial partition of D with mesh size $\eta > 0$, and let $\mathcal{T}_h(D)$ be a quasi-uniform refinement of $\mathcal{Y}$ with mesh size $\eta \geq h > 0$. If $\alpha \in L^\infty_+(D)$ is type-m quasi-monotone with respect to $\mathcal{Y}$ (in the sense of Definition 3.2) and X^* is a finite union of type-m facets F_j of the partition $\mathcal{Y}$ (not necessarily connected) such that $\overline{X}^* = \bigcup_{j\in\mathcal{J}} F_j$, then*

$$C^{*,m}_{P,\alpha} \lesssim \sigma^{d-m}\Big(\frac{\eta}{h}\Big) \frac{s_{\max}\, r_{\max}\, \eta^{m+2}}{\mathrm{meas}_m(X^*)\,\mathrm{diam}(Y)^2}$$

where $s_{\max}$ and $r_{\max}$ are defined as in Lemma 4.16 for type-$(d-1)$.

Proof. Recall the notation $\overline{u}^{X^*} = \frac{1}{\mathrm{meas}_0(X^*)} \sum_{j\in\mathcal{J}} u(F_j)$ introduced in Section 2.1 for the case $m = 0$, where $\mathrm{meas}_0(X^*) = \sum_{j\in\mathcal{J}} 1$ and F_j is a type-0 facet, i.e. a point. Similarly, we define $\int_{X^*} v\, ds := \sum_{j\in\mathcal{J}} v(p_j)$.

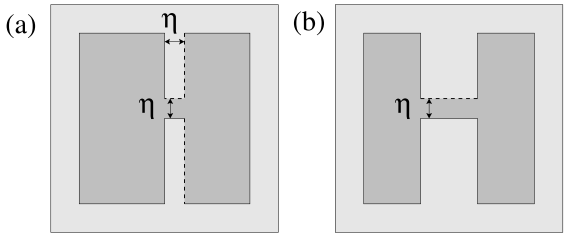


FIGURE 10. Two classes of "dumbbell" coefficient distributions. Dashed lines indicate variable interfaces for changing η.

With this notation, it is straightforward to follow the proof of Lemma 4.15 and to show that for any type-m quasi-monotone path $P_{k,j}$ from Y_k to Y_j, such that $F_j \subset \overline{Y}_j$, we have

$$\int_{F_j} \int_{Y_k} \left|u(x) - u(y)\right|^2 dy\, ds_x \;\lesssim\; s_{k,j}\, \eta^{m+2}\, \sigma^{d-m}\Big(\frac{\eta}{h}\Big)\, |u|^2_{H^1(P_{k,j})}\,.$$

The only difference is that we use Lemma 4.8 and Lemma 3.1 to prove the respective inequalities (4.10) and (4.11) for the (general) type-m case. The rest of the proof is analogous to that of Lemma 4.16. □

Example 4.19. In the 2D checkerboard example in Figure 9, we assume that the coefficient takes two values α_1 and $\alpha_2 \gg \alpha_1$. We choose X^* as the union of $\mathcal{O}(H/\eta)$ vertices on the boundary of D, as shown, and construct type-0 quasi-monotone paths $P_{k,j}$ from every square $Y_k \in \mathcal{Y}$ to every vertex $F_j \in X^*$, as shown in the figure. As in Example 4.17 and in Section 4.4, it is easy to see that these paths satisfy $s_{\max} \lesssim H/\eta$ and $r_{\max} \lesssim (H/\eta)^2$, and so, since $\mathrm{meas}_0(X^*) \eqsim H/\eta$, we finally get from Lemma 4.18 that

$$C^{*,m}_{P,\alpha} \;\lesssim\; \sigma^2\Big(\frac{\eta}{h}\Big)\, \frac{H}{\eta}\, \frac{H^2}{\eta^2}\, \frac{\eta^2}{H/\eta\, H^2} \;=\; 1 + \log\Big(\frac{\eta}{h}\Big)\,.$$

5. Numerical results

In this section we compute for some examples approximations of the weighted Poincaré constant $C_{P,\alpha}(D)$ by computing the smallest nonzero eigenvalue of the generalised eigenvalue problem

$$K_h\, \underline{u}_h \;=\; \lambda\, M_h\, \underline{u}_h\,.$$

Here K_h is the α-weighted stiffness matrix, M_h is the α-weighted mass matrix and $\underline{u}_h$ is the coefficient vector of the continuous, piecewise linear finite element approximation $u_h \in V_h(D)$ to the corresponding eigenfunction in (1.6)–(1.7) in Section 1 on a suitable mesh $\mathcal{T}_h(D)$. For the eigencomputations we have used the LOBPCG algorithm [18] with a factorisation of $(K_h{+}M_h)^{-1}$ as a preconditioner. For the latter we have used PARDISO [31, 32].

5.1. "Dumbbell"-type coefficients.

Here we study the two "dumbbell"-type coefficient distributions on $D = (0,1)^2$ shown in Figure 10. In each particular case, a suitable shape-regular partition $\{Y_\ell\}_{\ell=1}^n$ can be found such that the following holds:

Case (a): $s_{\max} \eqsim 1 + \log(H/\eta)$, and so Lemma 4.1 implies $C^*_P \lesssim 1 + \log(H/\eta)$.
Case (b): $s_{\max} \eqsim H/\eta$, and so $C^*_P \lesssim H/\eta$.

Figure 11 shows the approximate Poincaré constants for $\alpha = 10^5$ inside the dumbbell and $\alpha = 1$ otherwise. We used a uniform simplicial grid $\mathcal{T}_h(D)$ with $2 \times 512 \times 512$ elements. As we see our bounds are sharp and for the considered range of $\eta \in [\frac{1}{16}, \frac{1}{256}]$, the Poincaré constants are always bounded by 10 (even for Case (b)).

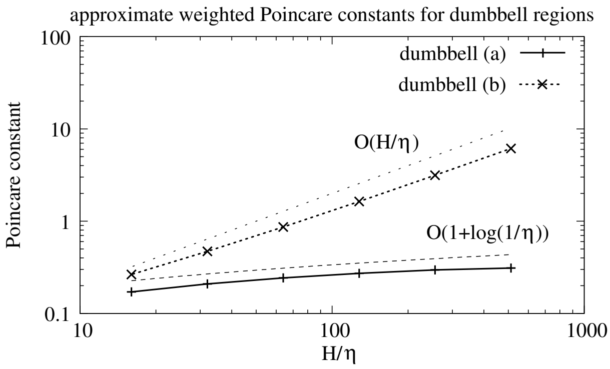


FIGURE 11. Approximate Poincaré constants for the dumbbell distributions in Figure 10(a)–(b) for different parameters η.

5.2. **Checkerboard distribution.** In Section 4.5 we have shown that in the case of the checkerboard distribution in Figure 9 the discrete weighted Poincaré constant in (4.4) can be bounded independent of α by

$$C_{P,\alpha}^{*,m} \lesssim 1 + \log\left(\frac{\eta}{h}\right).$$

We can observe this behaviour in Table 1 for the case $\alpha_1 = 1$ and $\alpha_2 = 10^5$. Keeping η fixed and decreasing h by a constant factor 1/2 each time, we see a constant additive growth in the Poincaré constant. Also, when η/h is constant, which corresponds to diagonals in the table, the Poincaré constant does not change significantly.

$\eta_{\min}$	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
$h = 1/4$	0.07344	–	–	–	–	–	–	–
1/8	0.1083	0.05777	–	–	–	–	–	–
1/16	0.1466	0.0799	0.05339	–	–	–	–	–
1/32	0.1852	0.1061	0.07223	0.05189	–	–	–	–
1/64	0.2240	0.1331	0.09518	0.06961	0.05125	–	–	–
1/128	0.2629	0.1604	0.1191	0.09146	0.06852	0.05095	–	–
1/256	0.3017	0.1876	0.1432	0.1143	0.08991	0.06802	0.05080	–
1/512	0.3406	0.2150	0.1674	0.1374	0.1123	0.08921	0.06778	0.05073

TABLE 1. Discrete weighted Poincaré constants for the checkerboard distribution for various choices of η and h.

5.3. **Layers.** To study the scenario in Figure 8 (middle), we choose $\Omega = (0, 1)^3$. For n layers (of equal width) we set α to $10^{5\frac{i-1}{n-1}}$ in the i-th layer, where $i = 1, \dots, n$. On a mesh with $32 \times 32 \times 32$ elements and varying n from 2 to 32, the computed weighted Poincaré constant is always 0.0337466, which illustrates that it is completely independent of the number of layers.

5.4. **Coefficients growing towards an edge.** Here we study Example 4.14, see also Figure 7 (right). We choose $\Omega = (0, 1)^3$ and let α grow towards the edge of the cube. Let η denote the smallest width of the region of the largest coefficient, as in Figure 7 (right). Figure 12 (left) shows the coefficient distribution for $\eta = 1/32$, whereas Figure 12 (right) shows an approximation of the second eigenfunction of (1.6)–(1.7) for a mesh of $32 \times 32 \times 32$ elements. The approximated Poincaré constants for a fixed mesh and varying η are displayed in Table 2.

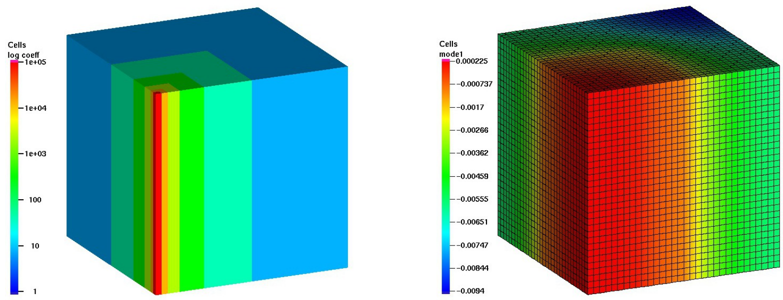


FIGURE 12. Coefficient distribution and second eigenfunction for Example 4.14 for $\eta = 1/32$ and $h = 1/32$.

η	1/4	1/8	1/16	1/32
$C_{P,\alpha}$	0.0588303	0.0637642	0.0700526	0.0764003

TABLE 2. Approximate Poincaré constants for Example 4.14 for the fixed mesh parameter $h = 1/32$.

APPENDIX A

Lemma A.1. *Let K be a (non-degenerate) d-dimensional simplex ($d = 2$ or 3) and let F be one of its facets. Then*

$$C_P(K;F) \;\le\; 1.$$

If K is a parallelepiped, then $C_P(K;F) \le 7/5$.

Proof. Veeser and Verfürth have shown that for all $v \in H^1(K)$:

$$\frac{1}{\mathrm{meas}_{d-1}(F)}\,\|v\|^2_{L^2(F)} \;\le\; \frac{1}{\mathrm{meas}_d(K)}\,\|v\|^2_{L^2(K)} + \frac{2\,\mathrm{diam}(K)}{\nu_K\,\mathrm{meas}_d(K)}\,\|v\|_{L^2(K)}\,|v|_{H^1(K)}\,, \tag{A.1}$$

where $\nu_K = d$ for the simplex and $\nu_K = 1$ for the parallelepiped. See [35, Sect. 4, Remark 4.6, formula (2.3), and Corollary 4.5]. Due to Payne & Weinberger [23] and Bebendorf [1],

$$\|u - \overline{u}^K\|_{L^2(K)} \;\le\; \frac{\mathrm{diam}(K)}{\pi}\,|u|_{H^1(K)} \qquad \forall u \in H^1(K), \tag{A.2}$$

because K is convex. With the triangle inequality and Cauchy's inequality,

$$\begin{aligned}
\|u - \overline{u}^F\|_{L^2(K)} \;&\le\; \|u - \overline{u}^K\|_{L^2(K)} + \sqrt{\mathrm{meas}_d(K)}\,|\overline{u}^K - \overline{u}^F| \\
&\le\; \|u - \overline{u}^K\|_{L^2(K)} + \frac{\sqrt{\mathrm{meas}_d(K)}}{\sqrt{\mathrm{meas}_{d-1}(F)}}\,\|u - \overline{u}^K\|_{L^2(F)}
\end{aligned}$$

Using (A.1) and (A.2) in the estimate above yields

$$\begin{aligned}
\|u - \overline{u}^F\|_{L^2(K)} \;&\le\; \tfrac{\mathrm{diam}(K)}{\pi}\,|u|_{H^1(K)} + \sqrt{\|u - \overline{u}^K\|^2_{L^2(K)} + \tfrac{2\,\mathrm{diam}(K)}{\nu_K}\,\|u - \overline{u}^K\|_{L^2(K)}\,|u|_{H^1(K)}} \\
&\le\; \tfrac{\mathrm{diam}(K)}{\pi}\,|u|_{H^1(K)} + \sqrt{\tfrac{\mathrm{diam}(K)^2}{\pi^2}\,|u|^2_{H^1(K)} + \tfrac{2\,\mathrm{diam}(K)}{\nu_K}\,\tfrac{\mathrm{diam}(K)}{\pi}\,|u|^2_{H^1(K)}} \\
&=\; \underbrace{\left(\frac{1}{\pi} + \sqrt{\frac{1}{\pi^2} + \frac{2}{\nu_K\,\pi}}\,\right)}_{:=C}\,\mathrm{diam}(K)\,|u|_{H^1(K)}
\end{aligned}$$

For the simplex $\nu_K = d \ge 2$ and we get $C \le 0.96609936 \le 1$. For the parallelepiped, $\nu_K = 1$ and so $C \le 1.17734478 \le \sqrt{7/5}$. $\square$

Lemma A.2. *Let T be a (non-degenerate) d-dimensional simplex ($d = 2$ or $d = 3$) and let $\rho(T)$ be the diameter of the largest inscribed ball in $\overline{T}$. Then*

$$\frac{\operatorname{diam}(T)^2}{\operatorname{meas}_d(T)} \;\leq\; 2^{d-1}\,\frac{\operatorname{diam}(T)}{\rho(T)^{d-1}}.$$

Proof. For $d = 2$, the estimate follows immediately from the well-known formula $\operatorname{meas}_2(T) = \frac{1}{4}\,\rho(T)\operatorname{meas}_1(\partial T) \geq \frac{1}{2}\,\rho(T)\operatorname{diam}(T)$ (recall that $\rho(T)$ is the *diameter* of the largest inscribed circle). For $d = 3$, we have $\operatorname{meas}_3(T) = \frac{1}{6}\,\rho(T)\operatorname{meas}_2(\partial T)$. Let $\{F_i\}_{i=1}^4$ be the four facets of T. Then, due to the two-dimensional formula above,

$$\operatorname{meas}_2(\partial T) \;\geq\; \sum_{i=1}^{4} \tfrac{1}{2}\,\rho(F_i)\operatorname{diam}(F_i).$$

Apparently $\rho(F_i) \geq \rho(T)$ for all $i = 1, \ldots, 4$. Moreover, the diameter of at least two faces equals $\operatorname{diam}(T)$ and the sum of the diameters of the remaining faces is at least $\operatorname{diam}(T)$, in other words $\sum_{i=1}^4 \operatorname{diam}(F_i) \geq 3\operatorname{diam}(T)$. Summarizing,

$$\frac{\operatorname{diam}(T)^2}{\operatorname{meas}_3(T)} \;=\; \frac{6\operatorname{diam}(T)^2}{\rho(T)\operatorname{meas}_2(\partial T)} \;\leq\; \frac{6\operatorname{diam}(T)^2}{\frac{3}{2}\,\rho(T)^2\operatorname{diam}(T)} \;=\; 4\,\frac{\operatorname{diam}(T)}{\rho(T)^2}\,.$$

□

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