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FIRST-ORDER CHARACTERIZATIONS OF METRIC SUBREGULARITY AND CALMNESS OF CONSTRAINT SET MAPPINGS

HELMUT GFRERER*

Abstract. A condition ensuring metric subregularity (respectively calmness) of general multifunctions between Banach spaces is derived. In finite dimensions this condition can be expressed in terms of a derivative which appears to be a combination of the coderivative and the contingent derivative. It is further shown that this sufficient conditions is in some sense the weakest possible first-order condition sufficient for subregularity. We extend this condition under the additional assumption that one part of the multifunction is known to be subregular in advance. Special attention is given to constraint systems as they occur in optimization.

Key words. Metric subregularity, calmness, multifunctions, constraint qualification

AMS subject classifications. 90C31, 26E25, 49J53

1. Introduction. For many optimization problems the constraints can be formulated as an inclusion of the abstract form

$$0 \in G(x), \tag{1.1}$$

where $G : X \rightrightarrows Y$ is a multifunction between Banach spaces X and Y.

An important special case, appearing in many applications, is given by constraint systems of the form

$$0 \in g(x) - C, \tag{1.2}$$

where $g : X \to Y$ is a mapping and $C \subset Y$ is a set, i.e. we have $G(x) = g(x) - C$. A prominent example of a constraint system of the form (1.2) is given by the constraints of a possibly infinite dimensional mathematical programming problem with $Y = \hat{Y} \times \mathbb{R}^m$, $C = \{0\} \times \mathbb{R}^m_-$ and $g = (P, h_1, \ldots, h_m)$, i.e.

$$\begin{aligned} P(x) &= 0 \\ h_i(x) &\le 0,\ i = 1, \ldots, m. \end{aligned} \tag{1.3}$$

In what follows we refer to a *smooth constraint system* in case of (1.2), when g is strictly differentiable at a point $\bar{x}$ under consideration and C is closed convex.

One of the central tasks in optimization is the development of first-order necessary conditions at a local minimizer $\bar{x}$, the existence of nondegenerate multipliers being related to the validity of some constraint qualification condition. Such a constraint qualification is for instance the property of metric subregularity:

Definition 1.1. *A multifunction $G : X \rightrightarrows Y$ acting between normed spaces X and Y is called* metrically subregular *at $(\bar{x}, \bar{y}) \in \operatorname{gph} G$, provided there exists a neighborhood U of $\bar{x}$ and a real number $\kappa > 0$ such that*

$$\mathrm{d}(x, G^{-1}(\bar{y})) \le \kappa \mathrm{d}(\bar{y}, G(x)), \quad \forall x \in U. \tag{1.4}$$

The metric subregularity property was introduced by Ioffe [10],[12] using the terminology "regularity at a point". The notation "metric subregularity" was suggested in [3]. As already noted in [13], the definition of metric subregularity in [3] is slightly different but equivalent to the definition 1.1.

Subregularity is a weaker condition than the more familar property of *metric regularity*, where the inequality (1.4) should hold not only for $\bar{y}$ but for all y belonging to some neighborhood of $\bar{y}$, i.e. there is also some neighborhood V of $\bar{y}$ such that

$$\mathrm{d}(x, G^{-1}(y)) \le \kappa \mathrm{d}(y, G(x)), \quad \forall (x, y) \in U \times V. \tag{1.5}$$

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For a survey on the theory of metric regularity and also on the related notions of *pseudo-Lipschitz continuity*, *Aubin property*, *Lipschitz-like property* and *openness with a linear rate* we refer to [12] and to the monographs [14], [17], [20].

It is well known [3] that metric subregularity of G at $(\bar{x}, \bar{y})$ is equivalent to calmness of the inverse multifunction G^{-1} at $(\bar{y}, \bar{x})$. A multifunction $S : Y \rightrightarrows X$ is said to be *calm* at $(\bar{y}, \bar{x}) \in \operatorname{gph} S$, if there exists $\kappa > 0$ along with some neighborhoods U of $\bar{x}$ such that

$$S(y) \cap U \subset S(\bar{y}) + \kappa \|y - \bar{y}\| \mathcal{B}_X, \ \forall y \in Y. \tag{1.6}$$

Usually the definition of calmness requires condition (1.6) to hold for y sufficiently close to $\bar{y}$, but it can be easily verified that this and our condition are equivalent.

In the special case that the set $S(\bar{y})$ reduces locally to a singleton and S is calm at $(\bar{y}, \bar{x})$, i.e. condition (1.6) is replaced by

$$S(y) \cap U \subset \bar{x} + \kappa \|y - \bar{y}\| \mathcal{B}_X, \ \forall y \in Y,$$

then S is called *locally upper Lipschitz*.

Besides its importance for deriving optimality conditions, metric subregularity respectively calmness also plays an important role in the theory of exact penalty functions, cf. [2],[15]. As pointed out in [13], subregularity is also an important tool for characterizing qualification conditions in the subdifferential calculus of nonsmooth analysis. In the recent paper [16] the convergence behavior of an algorithm for solving (1.1) was characterized by the calmness property.

Hence there is a growing interest in criteria for subregularity and calmness, respectively. We refer to the papers [5],[6],[7],[8],[13],[21]. An important subclass of multifunctions which are known to be metrically subregular at every point of its graph, is given by polyhedral multifunctions, i.e. multifunctions whose graph is the union of finitely many polyhedral sets. This result is due to Robinson [19]. An important special case of polyhedral multifunctions is given by linear systems, where subregularity is a consequence of Hoffman's error bound [9]. Some extensions to the infinite dimensional case are given in [1, Section 2.5.7].

The aim of this paper is to give *pointbased* criteria for metric subregularity of G, which are actually verifiable. For the sake of simplicity we restrict the presentation of our results to the case of metric subregularity. The notation "pointbased" means that the results are expressed in terms of derivative-like constructions at the reference point alone. We present a new first-order sufficient condition for subregularity of a general multifunctions G at $(\bar{x}, \bar{y})$. We also show that this sufficient condition is optimal in the following sense: If this sufficient condition is not fulfilled for a subregular mapping, we show that we can find a C^1-mapping h satisfying $h(\bar{x}) = 0$, $Dh(\bar{x}) = 0$ such that $G + h$ is no longer subregular at $(\bar{x}, \bar{y})$.

Note that the property of metric subregularity is not stable under smooth perturbation, even for convex multifunctions, as demonstrated by the following example:

Example 1.2. Let $G : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$, $G(x_1, x_2) := (x_2, -x_2) - \mathbb{R}^2_-$, $\bar{x} = \bar{y} = (0, 0)$. Then, $G^{-1}(\bar{y}) = \mathbb{R} \times \{0\}$ and therefore $\mathrm{d}((x_1, x_2), G^{-1}(\bar{y})) = |x_2| = \mathrm{d}(\bar{y}, G(x_1, x_2))$ holds for all $(x_1, x_2) \in \mathbb{R}^2$, showing G is metrically subregular at $(0, 0)$.

On the other hand, let $\varphi : \mathbb{R} \to \mathbb{R}$ denote the function fulfilling $\varphi(0) = \varphi'(0) = 0$, $\varphi''(t) = \exp(-t^{-2})$, $\forall t \neq 0$. Then φ is convex, $\varphi \in C^\infty(\mathbb{R})$, all derivatives $\varphi^{(i)}(0) = 0$, $i \in \mathbb{N}$ vanish at 0 and $\varphi(t) > 0$ holds for all $t \neq 0$. For arbitrarily fixed $\epsilon > 0$ let $\tilde{G}(x_1, x_2) := G(x_1, x_2) + (\epsilon\varphi(x_1), 0)$. Then $\tilde{G}^{-1}(\bar{y}) = \{\bar{x}\}$, implying $\mathrm{d}((x_1, 0), G^{-1}(\bar{y})) = |x_1|$. Since $\mathrm{d}(\bar{y}, \tilde{G}(x_1, 0)) = \epsilon\varphi(x_1)$ and $\lim_{t \to 0} |t|/\varphi(t) = \infty$, the perturbed multifunction $\tilde{G}$ is not subregular.

Hence we cannot expect that we can find an equivalent characterization of metric subregularity by means of a derivative-like object which is really computable in general, because any such derivative would have the property that the derivative of the multifunctions G and $\tilde{G}$ of example 1.2 are different. But G and $\tilde{G}$ differ only by a C^∞-function, where all derivatives vanish at $\bar{x}$ and so the usual calculus rules cannot be valid. Hence it is very difficult or even impossible to calculate such a derivative in general situations. Nevertheless, although such characterizations seem to be not verifiable they might be of interest in many other fields.

When our sufficient condition for subregularity is applied to closed convex multifunctions or smooth constraint systems, our sufficient condition holds if and only if either the "full" metrically regularity property holds or G^{-1} is locally upper Lipschitz. This means, if the solution set $G^{-1}(\bar{y})$ is not locally unique, in the smooth or in the convex case what we can show using exclusively some reasonable first-order theory, is the full metric regularity property. Hence we want to include some other kind of information such that we can also show subregularity when regularity does not hold. We give a sufficient condition for subregularity under the assumption that some part of the multifunction G is known to be subregular in advance, e.g. when we can identify some part of G which is polyhedral. In fact, if we consider again example 1.2 then $\operatorname{gph} G = ((x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x_2 - y_1 \leq 0, -x_2 - y_2 \leq 0\}$ and hence the multifunction G is polyhedral and consequently subregular.

For the reminder of the paper let X and Y be Banach spaces, $G : X \rightrightarrows Y$ denotes a multifunction and $(\bar{x}, \bar{y}) \in \operatorname{gph} G$ denotes a pair of points where we want to characterize subregularity of G. To ease the notation we assume $\bar{y} = 0$ in case of a constraint system (1.2), especially incase of a smooth constraint system.

In a normed space Z, $\mathcal{B}_Z := \{z \in Z \mid \|z\| \leq 1\}$ denotes the closed unit ball and $\mathcal{S}_Z := \{z \in Z \mid \|z\| = 1\}$ denotes the unit sphere. The topological dual space is denoted by Z^*. By $\langle z^*, z\rangle$ we denote the value $z^*(z)$ of the linear functional $z^* \in Z^*$ at $z \in Z$. For a set $D \subset Z$ we denote by $\sigma_D(\cdot)$ its support function, $\sigma_D(z^*) := \sup_{z \in D}\langle z^*, z\rangle$. $\operatorname{lin} D$ denotes the closed linear hull of D. Given $z \in D$ and $\epsilon \geq 0$ we denote by

$$\hat{N}_\epsilon(z; D) = \{z^* \in Z^* \mid \limsup_{u \xrightarrow{D} z} \frac{\langle z^*, u - z\rangle}{\|u - z\|} \leq \epsilon\}$$

the *set of* ϵ*-normals* to D. If $z \notin D$ we set $\hat{N}_\epsilon(z; D) = \emptyset$ for all $\epsilon \geq 0$. If D is convex, we denote the normal cone respectively the tangent cone at $z \in D$ by $N_D(z)$ and $T_D(z)$, respectively. Note that $N_D(z) = \hat{N}_0(z; D)$.

If not otherwise stated we endow the product space of normed spaces with the norm given by the sum of the individual norms.

The Fréchet (Gâteaux) derivative of a mapping g differentiable at $\bar{x}$ is denoted by $Dg(\bar{x})$. For a multifunction $S : Y \rightrightarrows X$ the *mixed coderivative* of S at $(\bar{y}, \bar{x}) \in \operatorname{gph} S$ is a multifunction $D^*_M S(\bar{y}, \bar{x}) : X^* \rightrightarrows Y^*$, where for each $x^* \in X^*$ the set $D^*_M S(\bar{y}, \bar{x})(x^*)$ is given by the collection of all $y^* \in Y^*$ for which there are sequences $(\epsilon_k) \downarrow 0$, $(y_k, x_k, x^*_k) \to (\bar{y}, \bar{x}, x^*)$ and $(y^*_k) \xrightarrow{w^*} y^*$ with $(y_k, x_k) \in \operatorname{gph} S$ and $(y^*_k, -x^*_k) \in \hat{N}_{\epsilon_k}((y_k, x_k), \operatorname{gph} S)$. Under a suitable sequential compactness assumption which e.g. is automatically fulfilled in finite dimensions, the mixed coderivative is an appropriate tool for characterizing metric regularity of a multifunction $G : X \rightrightarrows Y$: G is metrically regular near $(\bar{x}, \bar{y}) \in \operatorname{gph} G$ if and only if $D^*_M G^{-1}(\bar{y}, \bar{x})(0) = \{0\}$. For an exact statement of this and more far-leading properties the reader is referred to [17].

Another derivative known to be useful for characterizing subregularity is the *contingent derivative*, also called *graphical derivative* or *Bouligand derivative*. For a multifunction $G : X \rightrightarrows Y$ the contingent derivative at a point $(\bar{x}, \bar{y}) \in \operatorname{gph} G$ is the multifunction $CG(\bar{x}, \bar{y}) : X \rightrightarrows Y$ where for each $u \in X$ the set $CG(\bar{x}, \bar{y})(u)$ is given by the set of all $v \in Y$ for which there are sequences $(t_k) \downarrow 0$ and $(u_k, v_k) \to (u, v)$ with $(\bar{x} + t_k u_k, \bar{y} + t_k v_k) \in \operatorname{gph} G$. For a smooth constraint system the contingent derivative is given by $CG(\bar{x})(u) = Dg(\bar{x})u - T_C(g(\bar{x}))$.

It is well known (see e.g.[14]), that for finite dimensional spaces X and Y the solution mapping G^{-1} is locally upper Lipschitz at $(\bar{y}, \bar{x})$ if and only if $0 \notin CG(\bar{x}, \bar{y})(u)$, $\forall u \neq 0$. However the contingent derivative provides also the following necessary condition for metric subregularity:

Proposition 1.3. *Assume that the multifunction* $G : X \rightrightarrows Y$ *is metrically subregular at* $(\bar{x}, \bar{y}) \in \operatorname{gph} G$ *with some modulus* κ. *If*

1. *either* X *is finite dimensional,*
2. *or in case of a smooth constraint system, there exists a closed subspace* $\tilde{Y} \subset Y$ *such that* $C - g(\bar{x}) \subset \tilde{Y}$, *the subspace* $Dg(\bar{x})X - \tilde{Y}$ *is closed and* $Dg(\bar{x})(u + \mathcal{B}_X) \cap \tilde{Y}$ *is weakly compact for every* $u \in X$,

then the contingent derivative $CG(\bar x,\bar y)$ is metrically subregular at $(0,0)$ with modulus κ.

Proof. In order to prove the proposition we shall show that

$$\mathrm{d}(u, CG(\bar x)^{-1}(0)) \le \kappa \mathrm{d}(0, CG(\bar x)(u)) \tag{1.7}$$

holds for arbitrily fixed $u \in X$. This is certainly true if either $CG(\bar x)(u) = \emptyset$ or $0 \in CG(\bar x)(u)$. Hence we assume $0 \notin CG(\bar x)(u) \neq \emptyset$. Let $\epsilon > 0$ be arbitrarily fixed and let $v \in CG(\bar x)(u)$ be chosen such that $\|v\| < \mathrm{d}(0, CG(\bar x)(u)) + \epsilon$. Then there are sequences $(t_k) \downarrow 0$ and $(u_k, v_k) \to (u, v)$ such that $(\bar x + t_k u_k, \bar y + t_k v_k) \in \mathrm{gph}\, G$, implying $\mathrm{d}(\bar x + t_k u_k, G^{-1}(\bar y)) \le \kappa \mathrm{d}(\bar y, G(\bar x + t_k u_k)) \le \kappa t_k \|v_k\| \le \kappa t_k(\mathrm{d}(0, CG(\bar x)(u)) + \epsilon)$ for all k sufficiently large. Hence there exists a sequence $(\tilde u_k)$ such that $t_k \tilde u_k \in (G^{-1}(\bar y) - \bar x)$ and $\tilde u_k \in u_k + \kappa(\mathrm{d}(0, CG(\bar x)(u)) + 2\epsilon)\mathcal{B}_X$. In case that X is finite dimensional, by passing to a subsequence if necessary, $\tilde u_k$ converges to some $\bar u \in u + \kappa(\mathrm{d}(0, CG(\bar x)(u)) + 2\epsilon)\mathcal{B}_X$. Since $\bar y \in G(\bar x + t_k \tilde u_k)$ we deduce $0 \in CG(\bar x)(\bar u)$ showing $\mathrm{d}(u, CG(\bar x)^{-1}(0)) \le \|u - \bar u\| \le \kappa(\mathrm{d}(0, CG(\bar x)(u)) + 2\epsilon)$. Since this holds for arbitrary small ϵ, the bound (1.7) follows.

For smooth constraint systems we note that $\bar y = 0 \in G(\bar x + t_k \tilde u_k)$ implies $w_k \in Dg(\bar x)\tilde u_k - (C - g(\bar x))/t_k \subset Dg(\bar x)\tilde u_k - \tilde Y$, where $w_k := (g(\bar x) + t_k Dg(\bar x)\tilde u_k - g(\bar x + t_k \tilde u_k))/t_k = o(t_k)/t_k$. By the Generalized Open Mapping Theorem [18, Theorem 1] the multifunction $u \rightrightarrows Dg(\bar x)u - \tilde Y$ acting between X and the Banach space $Dg(\bar x)X - \tilde Y$ is metrically regular near $(0,0)$ with some modulus κ' and hence for each k sufficiently large there is some $\hat u_k \in X$ such that $-w_k \in Dg(\bar x)(\hat u_k - \tilde u_k) - \tilde Y$ and $\|\hat u_k - \tilde u_k\| \le \kappa' \|w_k\| = o(1)$. Therefore $0 \in Dg(\bar x)\hat u_k - \tilde Y$ and for all k sufficiently large we also have $\hat u_k \in u + \kappa(\mathrm{d}(0, CG(\bar x)(u)) + 3\epsilon)B_X$. By our compactness assumption and since for every weakly compact set K and every real λ the set λK is also weakly compact, there exists some $\bar u \in u + \kappa(\mathrm{d}(0, CG(\bar x)(u)) + 3\epsilon)B_X$ such that $Dg(\bar x)\bar u$ is a weak limit point of $Dg(\bar x)\hat u_k$. Since $\lim_k \mathrm{d}(Dg(\bar x)\hat u_k, T_C(g(\bar x))) = \lim_k \mathrm{d}(Dg(\bar x)\tilde u_k, T_C(g(\bar x))) = 0$ and $\mathrm{d}(\cdot, T_C(g(\bar x)))$ is a convex continuous and hence also weakly lower semicontiuous mapping, we conclude $\mathrm{d}(Dg(\bar x)\bar u, T_C(g(\bar x))) = 0$ and consequently $0 \in CG(\bar x)(\bar u)$, since $T_C(g(\bar x))$ is closed. This shows $\mathrm{d}(u, CG(\bar x)^{-1}(0)) \le \|u - \bar u\| \le \kappa(\mathrm{d}(0, CG(\bar x)(u)) + 3\epsilon)$ and again the bound (1.7) follows by taking $\epsilon \to 0$. □

Remark 1.4. In case of a smooth constraint system the assumptions are automatically fulfilled if one of the following two conditions holds:

1. X is reflexive. Then we can take $\tilde Y = Y$ and the compactness assumption follows from the weak continuity of $Dg(\bar x)$ and the weak compactness of the unit ball in reflexive Banach spaces.
2. $\tilde Y := \mathrm{lin}\,(C - g(\bar x))$ is finite dimensional and the subspace $Dg(\bar x)X$ is closed.

2. First-order conditions for subregularity.

DEFINITION 2.1. *We define the* critical limit set of G at $(\bar x,\bar y)$, *denoted by* $\mathrm{Cr}\, G(\bar x,\bar y)$, *as the set of all elements* $(v, x^*) \in Y \times X^*$ *such that there are sequences* $(t_k) \downarrow 0$, $(\epsilon_k) \downarrow 0$, $(v_k, x_k^*) \to (v, x^*)$, $(u_k, y_k^*) \subset \mathcal{S}_X \times \mathcal{S}_{Y^*}$ *with* $(-x_k^*, y_k^*) \in \hat N_{\epsilon_k}((\bar x + t_k u_k, \bar y + t_k v_k), \mathrm{gph}\, G)$.

The critical limit set is crucial for the characterization of subregularity. To keep the technical effort small we restrict our analysis of sufficient conditions to the case that the Banach space Y admits a Fréchet smooth renorm, i.e. there is an equivalent norm on Y that is Fréchet differentiable at any nonzero point. In particular, every reflexive space admits a Fréchet smooth renorm.

THEOREM 2.2.

1. *Assume that Y admits a Fréchet smooth renorm, that G has closed graph* $\mathrm{gph}\, G$ *and* $(0,0) \notin \mathrm{Cr}\, G(\bar x,\bar y)$. *Then G is metrically subregular at $(\bar x,\bar y)$.*
2. *If $(0,0) \in \mathrm{Cr}\, G(\bar x,\bar y)$ then there exists a continuously differentiable mapping $h : X \to Y$ with $h(\bar x) = 0$, $Dh(\bar x) = 0$ such that $G + h$ is not metrically subregular at $(\bar x,\bar y)$.*

Proof. For showing the first part we can assume without loss of generality that the original norm $\|\cdot\|$ on Y is Fréchet smooth since the properties metric subregularity and $(0,0) \in \mathrm{Cr}\, G(\bar x,\bar y)$ are invariant with respect to equivalent norms. We will show the first part by contradiction. Let us assume on the contrary that G is not metrically subregular at $(\bar x,\bar y)$. Then we can find for each k some element $x_k \in \bar x + \frac{1}{k}\mathcal{B}_X$ such that $\mathrm{d}(x_k, G^{-1}(\bar y)) > k\mathrm{d}(\bar y, G(x_k))$. We can choose some $y_k \in G(x_k)$ such that $\|\bar y - y_k\|$ approximates $\mathrm{d}(\bar y, G(x_k))$ sufficiently accurate so that $\mathrm{d}(x_k, G^{-1}(\bar y)) > k\|\bar y - y_k\|$ still holds. Since $\mathrm{d}(x_k, G^{-1}(\bar y)) > 0$ we obtain $x_k \notin G^{-1}(\bar y)$ and consequently $\|y_k - \bar y\| > 0$. Now let the function $\varphi : X \times Y \to \bar R$ be given by $\varphi(x,y) = \|y - \bar y\| + \chi_{\mathrm{gph}\, G}(x,y)$, where

$\chi_{\operatorname{gph} G}$ denotes the indicator function of $\operatorname{gph} G$. φ is l.s.c since $\|\cdot - \bar{y}\|$ is continuous and $\chi_{\operatorname{gph} G}$ is l.s.c due to the closed graph of G. By denoting $\epsilon = \|y_k - \bar{y}\|$ we obviously have $\epsilon = \varphi(x_k, y_k) \leq \inf_{X\times Y} \varphi(x, y) + \epsilon$ and so we can invoke Ekeland's variational principle to find some $(\tilde{x}_k, \tilde{y}_k)$ satisfying $\varphi(\tilde{x}_k, \tilde{y}_k) \leq \varphi(x_k, y_k)$, $\|(\tilde{x}_k, \tilde{y}_k) - (x_k, y_k)\| \leq \alpha$ and $\varphi(x, y) + \frac{\epsilon}{\alpha}\|(x, y) - (\tilde{x}_k, \tilde{y}_k)\| > \varphi(\tilde{x}_k, \tilde{y}_k)$ for all $(x, y) \neq (\tilde{x}_k, \tilde{y}_k)$, where $\alpha = \sqrt{k}\|y_k - \bar{y}\|$. From $\varphi(\tilde{x}_k, \tilde{y}_k) \leq \varphi(x_k, y_k) < \infty$ we deduce $(\tilde{x}_k, \tilde{y}_k) \in \operatorname{gph} G$. Using the triangle inequality we obtain

$$\text{(2.1)}\ \operatorname{d}(\tilde{x}_k, G^{-1}(\bar{y})) \geq \operatorname{d}(x_k, G^{-1}(\bar{y})) - \|\tilde{x}_k - x_k\| \geq \operatorname{d}(x_k, G^{-1}(\bar{y})) - \alpha > (k - \sqrt{k})\|y_k - \bar{y}\| \geq 0$$

showing $\tilde{x}_k \notin G^{-1}(\bar{y})$ and hence $\tilde{y}_k \neq \bar{y}$. Let $\tilde{y}_k^* \in Y^*$ denote the Fréchet derivative of $\|\cdot\|$ at $\tilde{y}_k - \bar{y}$, then $\tilde{y}_k^*$ belongs to the subdifferential of convex analysis and therefore $\tilde{y}_k^* \in \mathcal{S}_{Y^*}$ and $\langle \tilde{y}_k^*, \tilde{y}_k - \bar{y}\rangle = \|\tilde{y}_k - \bar{y}\|$. Due to the defintion of Fréchet differentiability we can find some positive δ_k such that $\|y - \bar{y}\| \leq \|\tilde{y}_k - \bar{y}\| + \langle \tilde{y}_k^*, y - \tilde{y}_k\rangle + \frac{1}{\sqrt{k}}\|y - \tilde{y}_k\|$ for all $y \in \tilde{y}_k + \delta_k\mathcal{B}_Y$. Hence for all $(x, y) \in \operatorname{gph} G \cap ((\tilde{x}_k, y_k) + \delta_k\mathcal{B}_{X\times Y})$ we have

$$\begin{aligned}\|\tilde{y}_k - \bar{y}\| = \varphi(\tilde{x}_k, \tilde{y}_k) &\leq \varphi(x, y) + \frac{1}{\sqrt{k}}\|(x, y) - (\tilde{x}_k, \tilde{y}_k)\| = \|y - \bar{y}\| + \frac{1}{\sqrt{k}}\|(x, y) - (\tilde{x}_k, \tilde{y}_k)\| \\ &\leq \|\tilde{y}_k - \bar{y}\| + \langle \tilde{y}_k^*, y - \tilde{y}_k\rangle + \frac{2}{\sqrt{k}}(\|x - \tilde{x}_k\| + \|y - \tilde{y}_k\|)\end{aligned}$$

showing $\langle -\tilde{y}_k^*, y - \tilde{y}_k\rangle \leq \frac{2}{\sqrt{k}}(\|x - \tilde{x}_k\| + \|y - \tilde{y}_k\|)$ and consequently $(0, -\tilde{y}_k^*) \in \hat{N}_{\frac{2}{\sqrt{k}}}((\tilde{x}_k, \tilde{y}_k), \operatorname{gph} G)$. Now set $\epsilon_k = \frac{2}{\sqrt{k}}$, $t_k = \|\tilde{x}_k - \bar{x}\|$, $u_k = t_k^{-1}(\tilde{x}_k - \bar{x})$, $v_k = t_k^{-1}(\tilde{y}_k - \bar{y})$, $x_k^* = 0$, $y_k^* = -\tilde{y}_k^*$. From (2.1) we obtain for $k \geq 2$ the inequality

$$\|\tilde{y}_k - \bar{y}\| \leq \|y_k - \bar{y}\| + \|\tilde{y}_k - y_k\| \leq (1 + \sqrt{k})\|y_k - \bar{y}\| \leq \frac{1 + \sqrt{k}}{k - \sqrt{k}}\operatorname{d}(\tilde{x}_k, G^{-1}(\bar{y})) \leq \frac{1 + \sqrt{k}}{k - \sqrt{k}}\|\tilde{x}_k - \bar{x}\|$$

implying $\|v_k\| \leq \frac{1+\sqrt{k}}{k-\sqrt{k}} \to 0$. Hence $(0, 0) \in \operatorname{Cr} G(\bar{x}, \bar{y})$, a contradiction.

To show the second part, let in accordance with definition 2.1 $(t_k, \epsilon_k, u_k, v_k, x_k^*, y_k^*)$ denote a sequence corresponding with the assumption $(0, 0) \in \operatorname{Cr} G(\bar{x}, \bar{y})$. By passing to a subsequence if necessary me may assume that $\|v^k\| \leq \frac{1}{k^2}$, $\|x_k^*\| \leq \frac{1}{2k}$, $\epsilon_k < \frac{1}{2k}$ and $t_{k+1} \leq \frac{t_k}{4}$ for all k. Setting $(x_k, y_k) = (\bar{x}, \bar{y}) + t_k(u_k, v_k)$ we can find for each k some positive radius $\rho_k \leq \frac{t_k}{2k}$ such that according to the condition $(-x_k^*, y_k^*) \in \hat{N}_{\epsilon_k}((\bar{x} + t_k u_k, \bar{y} + t_k v_k), \operatorname{gph} G)$ we have $\langle -x_k^*, x - x_k\rangle + \langle y_k^*, y - y_k\rangle \leq \frac{1}{2k}(\|x - x_k\| + \|y - y_k\|)$ for every $(x, y) \in ((x_k, y_k) + \rho_k\mathcal{B}_{X\times Y}) \cap \operatorname{gph} G$ and consequently

$$\begin{aligned}\text{(2.2)}\qquad \langle y_k^*, y - y_k\rangle &\leq \langle -x_k^*, x - x_k\rangle + \langle y_k^*, y - y_k\rangle + \frac{1}{2k}\|x - x_k\| \\ &\leq \frac{1}{k}(\|x - x_k\| + \|y - y_k\|), \quad \forall (x, y) \in ((x_k, y_k) + \rho_k\mathcal{B}_{X\times Y}) \cap \operatorname{gph} G\end{aligned}$$

Next we define for each k the function $\xi_k : X \to \mathbb{R}_+$ by

$$\xi_k(x) := 4t_k^{-2}(\langle q_k^*, x - x_k\rangle^2 + \sum_{i=1}^{k-1} 16^{i-k}\langle p_{ki}^*, x - x_k\rangle^2),$$

where the continuous linear functionals $q_k^* \in \mathcal{S}_{X^*}$, $p_{ki}^* \in \mathcal{S}_{X^*}$, $i < k$ are chosen such that

$$\langle q_k^*, \bar{x} - x_k\rangle = \|x_k - \bar{x}\| = t_k, \quad \langle p_{ki}^*, x_i - x_k\rangle = \|x_i - x_k\|,\ i < k.$$

We define the mapping $h : X \to Y$ by

$$h(x) := \sum_{k=1}^{\infty} h_k(x) := \sum_{k=1}^{\infty} -\max\{1 - \xi_k(x), 0\}^2(t_k v_k + \frac{\rho_k}{\sqrt{k}} z_k),$$

where the elements $z_k \in \mathcal{S}_Y$ are chosen such that $\langle y_k^*, z_k\rangle \geq 1/2$. h is well defined because Y is a Banach space and

$$\sum_{k=1}^{\infty} \|\max\{1-\xi_k(x),0\}^2(t_kv_k+\frac{\rho_k}{\sqrt{k}}z_k)\| \leq \sum_{k=1}^{\infty}\|t_kv_k+\frac{\rho_k}{\sqrt{k}}z_k\| \leq \sum_{k=1}^{\infty} t_k(k^{-2}+k^{-\frac{3}{2}}) < \infty. \tag{2.3}$$

Now let l and $x \in x_l + \frac{\rho_l}{2}\mathcal{B}_X$ be fixed and consider $\xi_k(x)$ for arbitrary k. If $l = k$ we obtain the bound

$$0 \leq \xi_k(x) \leq 4t_k^{-2}\|x-x_k\|^2(1+\sum_{i=1}^{k-1}16^{i-k}) \leq \frac{64}{15}t_k^{-2}\|x-x_k\|^2 \leq \frac{16}{15}t_k^{-2}\rho_k^2 \leq \frac{4}{15}k^{-2} \leq \frac{4}{15} < 1.$$

From these inequalities we can also deduce the estimates

$$0 \leq t_k(1-(1-\xi_k(x))^2) \leq 2t_k\xi_k(x) \leq \frac{32}{15}\rho_k\frac{\rho_k}{t_k} \leq \frac{16}{15}\frac{\rho_k}{k}, \quad \frac{1}{2} < (1-\xi_k(x))^2 \leq 1, \tag{2.4}$$

valid for all k and all $x \in x_k + \frac{\rho_k}{2}\mathcal{B}_X$.

If $l < k$ we have $\langle p_{kl}^*, x-x_k\rangle \geq \|x_l-x_k\| - \frac{\rho_l}{2} \geq t_l - t_k - \frac{t_l}{4l} \geq \frac{3}{4}t_l - t_k \geq (\frac{3}{4}4^{k-l}-1)t_k \geq \frac{1}{2}4^{k-l}t_k$ and consequently $\xi_k(x) \geq 1$. We also have $\xi_k(x) \geq 1$ when $l > k$ because of $\langle q_k^*, x-x_k\rangle = \langle q_k^*, \bar{x}-x_k\rangle + \langle q_k^*, x_l - \bar{x}\rangle + \langle q_k^*, x-x_l\rangle \geq t_k - t_l - \frac{t_l}{4l} \geq t_k - \frac{5}{4}t_l \geq \frac{11}{16}t_k > \frac{1}{2}t_k$. Hence we obtain

$$h(x) = -(1-\xi_k(x))^2(t_kv_k + \frac{\rho_k}{\sqrt{k}}z_k), \quad \forall k \forall x \in x_k + \frac{\rho_k}{2}\mathcal{B}_X.$$

Obviously $\xi_k(x_k) = 0$ and therefore $h(x_k) = -(t_kv_k + \frac{\rho_k}{\sqrt{k}}z_k)$. Using the fact $(x_k, y_k) \in \operatorname{gph} G$ we obtain

$$\mathrm{d}(\bar{y}, (G+h)(x_k)) \leq \|\bar{y} - (y_k + h(x_k))\| = \|\frac{\rho_k}{\sqrt{k}}z_k\| = \frac{\rho_k}{\sqrt{k}}. \tag{2.5}$$

Next we show that for all $k \geq 17$ we have $\mathrm{d}(x_k, (G+h)^{-1}(\bar{y})) \geq \frac{\rho_k}{2}$, i.e. $\bar{y} \notin (G+h)(x)$ for all $x \in x_k + \frac{\rho_k}{2}\mathcal{B}_X$. Assume on the contrary that $\bar{y} \in (G+h)(x)$ for some $k \geq 17$ and some $x \in x_k + \frac{\rho_k}{2}\mathcal{B}_X$. Then $\bar{y} - h(x) \in G(x)$ and by using (2.4) we obtain

$$\begin{aligned}\|\bar{y} - h(x) - y_k\| &= \|((1-\xi_k(x))^2-1)t_kv_k + (1-\xi_k(x))^2\frac{\rho_k}{\sqrt{k}}z_k\| \\ &\leq t_k(1-(1-\xi_k(x))^2)\|v_k\| + (1-\xi_k(x))^2\frac{\rho_k}{\sqrt{k}}\|z_k\| \leq \rho_k(\frac{16}{15k^3}+\frac{1}{\sqrt{k}}) < \frac{\rho_k}{2},\end{aligned}$$

implying $(x, \bar{y}-h(x)) \in \operatorname{gph} G \cap ((x_k,y_k) + \rho_k\mathcal{B}_{X\times Y})$. Hence we can use (2.2) to obtain

$$\langle y_k^*, \bar{y} - h(x) - y_k\rangle \leq \frac{1}{k}(\|x-x_k\| + \|\bar{y}-h(x)-y_k\|) \leq \frac{\rho_k}{k}$$

On the other hand, by using (2.4) again we have

$$\begin{aligned}\langle y_k^*, \bar{y}-h(x)-y_k\rangle &= \langle y_k^*, ((1-\xi_k(x))^2-1)t_kv_k + (1-\xi_k(x))^2\frac{\rho_k}{\sqrt{k}}z_k\rangle \\ &\geq -t_k(1-(1-\xi_k(x))^2)\|v_k\| + (1-\xi_k(x))^2\frac{\rho_k}{\sqrt{k}}\langle y_k^*, z_k\rangle \geq \rho_k(-\frac{16}{15k^3}+\frac{1}{4\sqrt{k}})\end{aligned}$$

and therefore $-\frac{16}{15k^3} + \frac{1}{4\sqrt{k}} \leq \frac{1}{k}$, a contradiction, because $\frac{1}{4\sqrt{k}} \geq \frac{1}{k} + \frac{16}{15k^3}$, $\forall k \geq 17$.

Hence $\mathrm{d}((x_k, (G+h)^{-1}(\bar{y})) \geq \frac{\rho_k}{2}$, $\forall k \geq 17$. Together with (2.5) we conclude that the elements of the sequence $(x_k) \to \bar{x}$ fulfill

$$\mathrm{d}(x_k, (G+h)^{-1}(\bar{y})) \geq \frac{\sqrt{k}}{2}\mathrm{d}(\bar{y}, (G+h)(x_k)), \quad \forall k \geq 17.$$

Hence $G + h$ is not metrically subregular at $(\bar{x}, \bar{y})$.

There remains to show that h is continuously differentiable and $h(\bar{x}) = 0$, $Dh(\bar{x}) = 0$. Continuity of h follows from uniform convergence of $\sum_k h_k(x)$ with respect to x due to (2.3) and the continuity of the functions h_k. Each of the functions h_k is also continuously differentiable with derivative

$$Dh_k(x)u = 2\max\{1 - \xi_k(x), 0\}\langle D\xi_k(x), u\rangle(t_k v_k + \frac{\rho_k}{\sqrt{k}} z_k),$$

where

$$\langle D\xi_k(x), u\rangle = 8t_k^{-2}(\langle q_k^*, x - x_k\rangle\langle q_k^*, u\rangle + \sum_{i=1}^{k-1} 16^{i-k}\langle p_{ki}^*, x - x_k\rangle\langle p_{ki}^*, u\rangle).$$

Since $q_k^*, p_{ki}^* \in S_{X^*}$ we obtain by using the Cauchy–Schwarz inequality

$$\begin{aligned}
\frac{t_k^2}{8}\|D\xi_k(x)\| &\le |\langle q_k^*, x - x_k\rangle| + \sum_{i=1}^{k-1} 16^{i-k}|\langle p_{ki}^*, x - x_k\rangle| \\
&\le (1 + \sum_{i=1}^{k-1} 16^{i-k})^{\frac{1}{2}}(\langle q_k^*, x - x_k\rangle^2 + \sum_{i=1}^{k-1} 16^{i-k}\langle p_{ki}^*, x - x_k\rangle^2)^{\frac{1}{2}} \\
&\le t_k\sqrt{\xi_k(x)}
\end{aligned}$$

and consequently

$$\begin{aligned}
\sum_{k=1}^{\infty}\|Dh_k(x)\| &\le \sum_{k=1}^{\infty} 2\max\{1 - \xi_k(x), 0\}\|D\xi_k(x)\|\|t_k v_k + \frac{\rho_k}{\sqrt{k}} z_k\| \\
&\le \sum_{k=1}^{\infty} 2\max\{1 - \xi_k(x), 0\} 8t_k^{-1}\sqrt{\xi_k(x)}t_k(\|v_k\| + \frac{\rho_k}{t_k}\frac{\|z_k\|}{\sqrt{k}}) \\
&\le \sum_{k=1}^{\infty} 16(\frac{1}{k^2} + \frac{1}{2k^{\frac{3}{2}}}) < \infty
\end{aligned}$$

for all $x \in X$. Hence h is continuously differentiable owing to the uniform convergence of $\sum_k Dh_k(x)$ with respect to x.

Finally we have $\langle q_k^*, \bar{x} - x_k\rangle = t_k$, $\forall k$ by the definition of q_k^* and therefore $\xi_k(\bar{x}) \ge 4$. Hence, $\max\{1 - \xi_k(\bar{x}), 0\} = 0$ and $h(\bar{x}) = 0$ and $Dh(\bar{x}) = 0$ follows. This completes the proof of the theorem.□

The condition $(0, 0) \in \operatorname{Cr} G(\bar{x}, \bar{y})$ is related with the following derivative:

Definition 2.3. *We define the* combined contingent coderivative of G at $(\bar{x}, \bar{y})$ *as the multifunction* $CD^*G(\bar{x}, \bar{y}) : X \times Y^* \rightrightarrows Y \times X^*$, *where for each* $(u, y^*) \in X \times Y^*$ *the set* $CD^*G(\bar{x}, \bar{y})(u, y^*)$ *is given by the collection of all* $(v, x^*) \in Y \times X^*$ *for which there are sequences* $(t_k) \downarrow 0$, $(\epsilon_k) \downarrow 0$, $(u_k, v_k, x_k^*) \to (u, v, x^*)$, $(y_k^*) \stackrel{w^*}{\to} y^*$ *with* $(-x_k^*, y_k^*) \in \hat{N}_{\epsilon_k}((\bar{x} + t_k u_k, \bar{y} + t_k v_k), \operatorname{gph} G)$.

By the definition the elements $(v, x^*) \in CD^*G(\bar{x}, \bar{y})(u, y^*)$ fulfill the relations $v \in CG(\bar{x}, \bar{y})$ and $y^* \in D_M^*G^{-1}(\bar{y}, \bar{x})(x^*)$. In fact, the combined contingent coderivative of G is defined by elements of the contingent derivative of G and the mixed coderivative of G^{-1} which share in their definition a common sequence of points $(\bar{x} + t_k u_k, \bar{y} + t_k v_k)$.

We always have the implication

$$\Big(\exists (u, y^*) \in (X \setminus \{0\}) \times (Y^* \setminus \{0\}) : (0, 0) \in CD^*G(\bar{x}, \bar{y})(u, y^*)\Big) \;\Rightarrow\; (0, 0) \in \operatorname{Cr} G(\bar{x}, \bar{y})$$

In fact, in accordance with the definition let us choose some sequence $(t_k, \epsilon_k, u_k, v_k, x_k^*, y_k^*)$ corresponding to the condition $(0, 0) \in CD^*G(\bar{x}, \bar{y})(u, y^*)$, where $\|u\| \cdot \|y^*\| \ne 0$. By the convergence of u_k and the weak-* convergence of y_k^* to non-zero elements we can assume the existence of some positive α such that for all k we have $\frac{1}{\alpha} \geq \|u_k\| \geq \alpha$ and $\|y_k^*\| \geq \alpha$, respectively. Setting

$$(\tilde{t}_k, \tilde{\epsilon}_k, \tilde{u}_k, \tilde{v}_k, \tilde{x}_k^*, \tilde{y}_k^*) = (t_k\|u_k\|, \frac{\epsilon_k}{\|y_k^*\|}, \frac{u_k}{\|u_k\|}, \frac{v_k}{\|u_k\|}, \frac{x_k^*}{\|y_k^*\|}, \frac{y_k^*}{\|y_k^*\|})$$

we have $\tilde{t}_k \downarrow 0, \tilde{\epsilon}_k \downarrow 0$, $(\tilde{v}_k, \tilde{x}_k^*) \to (0,0)$, $(\tilde{u}_k, \tilde{y}_k^*) \in \mathcal{S}_X \times \mathcal{S}_{Y^*}$ and $(-\tilde{x}_k^*, \tilde{y}_k^*) \in \hat{N}_{\tilde{\epsilon}_k}((\bar{x} + \tilde{t}_k\tilde{u}_k, \bar{y} + \tilde{t}_k\tilde{v}_k), \operatorname{gph} G)$, showing $(0,0) \in \operatorname{Cr} G(\bar{x}, \bar{y})$. The reverse implication is also true under some sequential compactness assumption:

Assumption 1. *For every sequence* $\zeta_k = (t_k, \epsilon_k, u_k, v_k, x_k^*, y_k^*) \in \mathbb{R}_+ \times \mathbb{R}_+ \times X \times Y \times X^* \times Y^*$ *such that* $(t_k) \downarrow 0$*,* $(\epsilon_k) \downarrow 0$*,* $(v_k, x_k^*) \to (0,0)$*,* $(u_k, y_k^*) \in \mathcal{S}_X \times \mathcal{S}_{Y^*}$*,* $(-x_k^*, y_k^*) \in \hat{N}_{\epsilon_k}((\bar{x} + t_k u_k, \bar{y} + t_k v_k), \operatorname{gph} G)$ *there is a subsequence* (ζ_{k_n}) *such that* (u_{k_n}) *is convergent and* $(y_{k_n}^*)$ *is weak-* convergent to some nonzero element.*

Assumption 1 is always fulfilled if both X and Y are finite dimensional. On the other hand, from the second part of theorem 2.2 we can conclude that Assumption 1 is fulfilled for multifunctions G where the property of metric subregularity is stable under smooth perturbations h with $h(\bar{x}) = 0$, $Dh(\bar{x}) = 0$.

The considerations above show the following result:

Proposition 2.4. *If Assumption 1 is fulfilled, then the following statements are equivalent:*

1. $(0,0) \in \operatorname{Cr} G(\bar{x}, \bar{y})$
2. $(0,0) \in CD^*G(\bar{x}, \bar{y})(u, y^*)$ *for some* $u \neq 0$*,* $y^* \neq 0$*.*

It is easy to see that for a continuously differentiable mapping $h : X \to Y$ with $h(\bar{x}) = 0$, $Dh(\bar{x}) = 0$ we have $\operatorname{Cr} G(\bar{x}, \bar{y}) = \operatorname{Cr} (G + h)(\bar{x}, \bar{y})$. In fact, if $(-x^*, y^*) \in \hat{N}_\epsilon((x,y), \operatorname{gph} G)$ then $(-x^*, y^*) \in \hat{N}_{\epsilon + \|Dh(x)\|}((x, y + h(x)), \operatorname{gph}(G+h))$ and hence $\operatorname{Cr} G(\bar{x}, \bar{y}) \subset \operatorname{Cr}(G+h)(\bar{x}, \bar{y})$ follows. Then we also have $\operatorname{Cr}(G+h)(\bar{x}, \bar{y}) \subset \operatorname{Cr}(G+h-h)(\bar{x}, \bar{y})$ and therefore the desired equality follows. The same arguments also show that $CD^*G(\bar{x}, \bar{y}) = CD^*(G+h)(\bar{x}, \bar{y})$.

Theorem 2.2 tells us that the condition $(0,0) \notin \operatorname{Cr} G(\bar{x}, \bar{y})$ is the best possible sufficient condition for subregularity, as long as we restrict ourselves to derivative-like objects for multifunctions which are invariant under sufficiently small smooth perturbations. As an example let us consider a result due to Henrion, Jourani and Outrata [6, Theorem 3.2]: Given a multifunction $M : \mathbb{R}^p \rightrightarrows \mathbb{R}^k$ defined as the intersection $M(y) = S(y) \cap C$, where $S : \mathbb{R}^p \rightrightarrows \mathbb{R}^k$ is a multifunction with closed graph and $C \subset \mathbb{R}^k$ is closed and fulfills some additional regularity assumptions, then it is shown that M is calm at $(\bar{y}, \bar{x}) \in \operatorname{gph} M$, if for all $y^* \in \mathbb{R}^p$ it holds that

$$D^*S^{-1}(\bar{x}, \bar{y})(y^*) - \operatorname{bd} N_C(\bar{x}) = \begin{cases} \emptyset & \text{or} \\ \{0\} & \text{if } y^* = 0 \end{cases},$$

where D^*S^{-1} denotes the coderivative of S^{-1} and N_C denotes the normal cone in the sense of Clarke. In our setting we study subregularity of the multifunction

$$G(x) := M^{-1}(x) = \begin{cases} S^{-1}(x) & \text{if } x \in C \\ \emptyset & \text{otherwise} \end{cases}$$

Now it follows from the second part of theorem 2.2 that the calmness criterion above implies our sufficient condition $(0,0) \notin \operatorname{Cr} G(\bar{x}, \bar{y})$, since for any C^1 function $h : \mathbb{R}^k \to \mathbb{R}^p$ with $h(\bar{x}) = 0$, $Dh(\bar{x}) = 0$ we have

$$(G+h)(x) = \begin{cases} (S^{-1} + h)(x) & \text{if } x \in C \\ \emptyset & \text{otherwise} \end{cases}$$

and $D^*S^{-1}(\bar{x}, \bar{y}) = D^*(S^{-1} + h)(\bar{x}, \bar{y})$.

In the literature there exist also characterizations for subregularity which are based on derivatives which are not invariant under smooth perturbations having 0 function value and derivative at the reference point. As an example let us mention here the *outer coderivative* as introduced in [13]. Using this outer coderivative one can even show that the multifunction G of example

1.2 is subregular, but, as already mentioned in the introduction, this also implies that the outer coderivative is hardly manageable. Let us mention that it is also easily possible to strengthen our condition $(0,0) \notin \mathrm{Cr}\, G(\bar{x},\bar{y})$, e.g. by requiring in the definition of the critical limit set the addditional condition $\bar{y} \notin G(\bar{x}+t_k u_k)$. But similar to the case of the outer coderivative this results in a criterion which is not invariant under smooth perturbations with 0 function value and derivative.

Example 2.5. Let $G : \mathbb{R} \rightrightarrows \mathbb{R}$ be given by $G(x) = \{g(x)\}$, where the continuous mapping $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = x$ for $x \notin (0,1]$, $g(x) = -2^{-(2k+1)} + 3|x - 2^{-(2k+1)}|$ for $x \in (2^{-(2k+2)}, 2^{-2k}]$, $k = 0,1,2,\ldots$. Note that $g(2^{-i}) = (-1)^i 2^{-i}$ holds for $i = 0,1,\ldots$. It is easily verified that G is metrically subregular at $(0,0)$, but G is neither metrically subregular near $(0,0)$ nor locally upper Lipschitz. Straightforward calculations yield that

$$\hat{N}_\epsilon((x,g(x)),\mathrm{gph}\, G) = \begin{cases} \{(x^*,y^*)\,|\,|y^*+x^*| \le 2\epsilon\} & \text{if } x<0\\ \{(x^*,y^*)\,|\,|y^*+x^*| \le 2\epsilon, x^* \le \epsilon, -y^*+x^* \le 2\epsilon\} & \text{if } x=0\\ \{(x^*,y^*)\,|\,|3y^*+x^*| \le 4\epsilon\} & \text{if } x \in (2^{-(2k+1)},2^{-2k})\\ \{(x^*,y^*)\,|\,|-3y^*+x^*| \le 4\epsilon\} & \text{if } x \in (2^{-(2k+2)},2^{-(2k+1)})\\ \{(x^*,y^*)\,|\,|x^*| \le 3y^*+4\epsilon\} & \text{if } x = 2^{-(2k+2)}\\ \{(x^*,y^*)\,|\,|x^*| \le -3y^*+4\epsilon\} & \text{if } x = 2^{-(2k+1)}. \end{cases}$$

It follows that $\mathrm{Cr}\, G(0,0) = ([-1,1]\times\{\pm 3\}) \cup (\{\pm 1\}\times[-3,3])$ and

$$CD^*G(0,0)(u,y^*) = \begin{cases} (u,y^*) & \text{if } u<0,\ y^*\neq 0\\ ([-u,u]\times\{\pm 3y^*\}) \cup (\{u\}\times[-3y^*,3y^*]) \cup (\{-u\}\times[3y^*,-3y^*]) & \text{if } u>0,\ y^*\neq 0. \end{cases}$$

Hence we can also verify the property of metric subregularity of G according to our theory.

PROPOSITION 2.6. *Assume that either G^{-1} is locally upper Lipschitz at $(\bar{y},\bar{x})$ or G is metrically regular near $(\bar{x},\bar{y})$. Then $(0,0) \notin \mathrm{Cr}\, G(\bar{x},\bar{y})$.*

Proof. By contraposition. Consider the sequence $(t_k,\epsilon_k,u_k,v_k,x_k^*,y_k^*)$ according to the condition $(0,0) \in \mathrm{Cr}\, G(\bar{x},\bar{y})$. Then we have $x_k := \bar{x}+t_k u_k \in G^{-1}(y_k)$, where $y_k := \bar{y}+t_k v_k$ and hence $\|x_k-\bar{x}\| = t_k = \frac{1}{\|v_k\|}\|y_k-\bar{y}\|$. Since $x_k \to \bar{x}$ and $v_k \to 0$, G^{-1} is not locally upper Lipschitz at $(\bar{y},\bar{x})$. Now assume on the contrary that G is metrically regular near $(\bar{x},\bar{y})$ and let U, V and κ be given in accordance to (1.5). Without loss of generality we can assume $\kappa > 1$ and by passing to a subsequence we can assume $\|x_k^*\| \le \frac{1}{k}$. For every k sufficiently large there is some positive δ_k such that $(x_k,y_k)+\delta_k\mathcal{B}_{X\times Y} \subset U\times V$,

$$\langle -x_k^*, x-x_k\rangle + \langle y_k^*, y-y_k\rangle \le (\epsilon_k+\frac{1}{k})(\|x-x_k\|+\|y-y_k\|) \tag{2.6}$$

holds for all $(x,y) \in ((x_k,y_k)+\delta_k\mathcal{B}_{X\times Y})\cap \mathrm{gph}\, G$. Then for every $h \in \mathcal{S}_Y$ we have $\mathrm{d}(x_k, G^{-1}(y_k+\frac{\delta_k}{2\kappa}h)) \le \kappa \mathrm{d}(y_k+\frac{\delta_k}{2\kappa}h, G(x_k)) \le \frac{\delta_k}{2}$ and hence there is some $x \in x_k+\frac{\delta_k}{2}\mathcal{B}_X$ such that $(x, y_k+\frac{\delta_k}{2\kappa}h) \in ((x_k,y_k)+\delta_k\mathcal{B}_{X\times Y})\cap \mathrm{gph}\, G$ and from (2.6) we obtain

$$\begin{aligned} \frac{\delta_k}{2\kappa}\langle y_k^*, h\rangle &= \langle y_k^*, y_k+\frac{\delta_k}{2\kappa}h-y_k\rangle\\ &\le (\epsilon_k+\frac{1}{k})(\|x-x_k\|+\|y_k+\frac{\delta_k}{2\kappa}h-y_k\|)+\|x_k^*\|\|x-x_k\|\\ &\le (\epsilon_k+\frac{1}{k})\delta_k+\frac{\delta_k}{2k}. \end{aligned}$$

This implies $\langle y_k^*, h\rangle \le \kappa(2\epsilon_k+\frac{3}{k})$, $\forall h \in \mathcal{S}_Y$ and consequently we obtain $\|y_k^*\| \to 0$, a contradiction to $y_k^* \in \mathcal{S}_{Y^*}$. □

PROPOSITION 2.7. *In case a smooth constraint system or alternatively, if G has closed convex graph, the following statements are equivalent:*

1. $(0,0) \notin \operatorname{Cr} G(\bar x,\bar y)$.
2. *G^{-1} is locally upper Lipschitz or G is metrically regular near $(\bar x,\bar y)$.*

Proof. In view of proposition 2.6 it suffices to show that $(0,0)\in \operatorname{Cr} G(\bar x,\bar y)$ whenever neither G^{-1} is locally upper Lipschitz nor G is metrically regular near $(\bar x,\bar y)$. Assuming G^{-1} not to be locally upper Lipschitz at $(\bar x,\bar y)$ there is a sequence $(\tilde x_k,\tilde y_k)\in \operatorname{gph} G$ such that $\tilde x_k$ converges to $\bar x$ and $\|\tilde x_k-\bar x\| > k\|\tilde y_k-\bar y\|$, $\forall k$. Setting $\tilde t_k=\|\tilde x_k-\bar x\|$ we have $\tilde t_k\downarrow 0$. Now let us consider the case that G has closed convex graph. Assuming G not to be metrically regular we have $\bar y\notin \operatorname{int}\operatorname{Range} G$ by the Robinson-Ursescu theorem (see, e.g. [1]). Since $\bar y\in\operatorname{Range} G$ there is a sequence $(\hat y_k)\subset Y\setminus \operatorname{cl}\operatorname{Range} G$ satisfying $\|\hat y_k-\bar y\|\le \frac{\tilde t_k}{k}$. We can separate $\hat y_k$ from $\operatorname{cl}\operatorname{Range} G$ by some linear functional $y_k^*\in\mathcal S_{Y^*}$, i.e. $\langle y_k^*,y\rangle\le\langle y_k^*,\hat y_k\rangle$, $\forall y\in \operatorname{cl}\operatorname{Range} G$. Hence $\langle -y_k^*,\tilde y_k\rangle\le \inf\{\langle -y_k^*,y\rangle\,|\,(x,y)\in\operatorname{gph} G\}+\langle -y_k^*,\tilde y_k-\hat y_k\rangle$. By Ekeland's variational principle we can find some $(x_k,y_k)\in\operatorname{gph}G$, $\|x_k-\tilde x_k\|+\|y_k-\tilde y_k\|\le\sqrt{k}\langle -y_k^*,\tilde y_k-\hat y_k\rangle\le\sqrt{k}(\|\tilde y_k-\bar y\|+\|\hat y_k-\bar y\|)\le\frac{2\tilde t_k}{\sqrt{k}}$, such that

$$\langle -y_k^*,y_k\rangle\le\langle -y_k^*,y\rangle+\frac{1}{\sqrt{k}}(\|x-x_k\|+\|y-y_k\|),\forall (x,y)\in\operatorname{gph}G$$

and consequently $(0,y_k^*)\in\hat N_{\frac{1}{\sqrt{k}}}((x_k,y_k),\operatorname{gph}G)$. Setting $\epsilon_k=1/\sqrt{k}$, $t_k=\|x_k-\bar x\|$, $u_k=(x_k-\bar x)/t_k$, $v_k=(y_k-\bar y)/t_k$, and $x_k^*=0$ we can conclude that $(0,0)\in\operatorname{Cr}G(\bar x,\bar y)$, since $|t_k-\tilde t_k|\le\|x_k-\tilde x_k\|\le 2\tilde t_k/\sqrt{k}$ implying $t_k\downarrow 0$ and

$$\|v_k\|\le\frac{\|y_k-\tilde y_k\|+\|\tilde y_k-\bar y\|}{t_k}\le\frac{\frac{2\tilde t_k}{\sqrt{k}}+\frac{\tilde t_k}{k}}{\tilde t_k(1-\frac{2}{\sqrt{k}})}\to 0.$$

Now assume that $G(x)=g(x)-C$ is not metrically regular at $(\bar x,0)$. It follows from [4] that $\sup_{y^*\in\mathcal S_{Y^*}}\langle y^*,g(\bar x)\rangle-\sigma_C(y^*)-\|Dg(\bar x)^*y^*\|\ge 0$. Hence there is a sequence $(\tilde y_k^*)\subset\mathcal S_{Y^*}$ satisfying $\|Dg(\bar x)^*\tilde y_k^*\|\le 1/k$, $\sigma_C(\tilde y_k^*)-\langle\tilde y_k^*,g(\bar x)\rangle\le\tilde t_k/k$ for all k. Since g is strictly differentiable at $\bar x$, by passing to a subsequence if necessary, we can also assume that

$$\eta_k:=\sup\{\frac{\|g(x)-g(\tilde x_k)-Dg(\bar x)(x-\tilde x_k)\|}{\|x-\tilde x_k\|}\,|\,x\in\tilde x_k+\tilde t_k\mathcal B_X, x\neq\tilde x_k\}\le\frac{1}{k}.$$

Denoting $\tilde c_k:=g(\tilde x_k)-\tilde y_k\in C$ and $y_k^*:=-\tilde y_k^*$ we obtain

$$\begin{aligned}\langle y_k^*,\tilde c_k\rangle&\le-\sigma_C(\tilde y_k^*)+\langle\tilde y_k^*,g(\bar x)-\tilde c_k\rangle+\frac{\tilde t_k}{k}\\&=-\sigma_C(\tilde y_k^*)+\langle\tilde y_k^*,g(\bar x)+Dg(\bar x)(\tilde x_k-\bar x)-g(\tilde x_k)+\tilde y_k\rangle-\langle Dg(\bar x)^*\tilde y_k^*,\tilde x_k-\bar x\rangle+\frac{\tilde t_k}{k}\\&\le-\sigma_C(-y_k^*)+\eta_k\|\tilde x_k-\bar x\|+\frac{3\tilde t_k}{k}\\&\le\inf_{c\in C}\langle y_k^*,c\rangle+\frac{4\tilde t_k}{k}.\end{aligned}$$

By Ekeland's variational principle we can find some $c_k\in C$ such that $\|c_k-\tilde c_k\|\le\frac{4\tilde t_k}{\sqrt{k}}$ and

$$\langle y_k^*,c_k\rangle=\inf_{c\in C}\{\langle y_k^*,c\rangle+\frac{1}{\sqrt{k}}\|c-c_k\|\}. \tag{2.7}$$

Now we define sequences $t_k:=\tilde t_k$, $x_k:=\tilde x_k$, $u_k:=(x_k-\bar x)/t_k$, $y_k:=g(x_k)-c_k$, $v_k:=y_k/t_k$, and $x_k^*:=0$. Then we obtain

$$\|v_k\|=\frac{\|g(x_k)-c_k\|}{t_k}\le\frac{\|g(x_k)-\tilde c_k\|}{t_k}+\frac{4}{\sqrt{k}}=\frac{\|\tilde y_k\|}{t_k}+\frac{4}{\sqrt{k}}\le\frac{1}{k}+\frac{4}{\sqrt{k}}$$

and therefore $v_k \to 0$. For every $(x,y) \in \operatorname{gph} G \cap ((x_k,y_k)+t_k\mathcal{B}_{X\times Y})$ we have $c := g(x)-y \in C$ and $\|c-c_k\| \le \|g(x)-y-g(x_k)+y_k\| \le (\|Dg(\bar x)\|+\eta_k+1)(\|x-x_k\|+\|y-y_k\|) \le (\|Dg(\bar x)\|+\eta_k+1)t_k$. Using (2.7) we obtain

$$\begin{aligned}
\langle y_k^*, y-y_k\rangle &= \langle y_k^*, g(x)-g(x_k)-(c-c_k)\rangle\\
&\le \langle y_k^*, Dg(\bar x)(x-\bar x)\rangle + \eta_k\|x-x_k\| + \frac{\|c-c_k\|}{\sqrt{k}}\\
&\le (\frac{1}{k}+\eta_k)\|x-x_k\| + \frac{(\|Dg(\bar x)\|+\eta_k+1)(\|x-x_k\|+\|y-y_k\|)}{\sqrt{k}}\\
&\le (\frac{1}{k}+\eta_k+\frac{(\|Dg(\bar x)\|+\eta_k+1)}{\sqrt{k}})(\|x-x_k\|+\|y-y_k\|)\\
&=: \epsilon_k(\|x-x_k\|+\|y-y_k\|)
\end{aligned}$$

showing $(-x_k^*, y_k^*) \in \hat N_{\epsilon_k}(\operatorname{gph} G, (x_k,y_k))$. Since $\epsilon_k \to 0$ we have established $(0,0) \in \operatorname{Cr} G(\bar x,\bar y)$ and this completes the proof. □

3. First order characterizations of subregularity under additional information. We consider now an extension of the sufficient condition of theorem 2.2 under the additional information that some part of G is known to be metrically subregular in advance.

DEFINITION 3.1. *For a subspace $Z \subset Y$ we define the critical limit set of G at $(\bar x,\bar y)$ with respect to Z as the set $\operatorname{Cr}_Z G(\bar x,\bar y)$ of all elements $(v,x^*) \in Y\times X^*$ such that there are sequences $(t_k)\downarrow 0$, $(\epsilon_k)\downarrow 0$, $(v_k,x_k^*) \to (v,x^*)$, $(u_k,y_k^*) \in \mathcal{S}_X\times\mathcal{S}_{Y^*}$ and a positive constant α with $(-x_k^*,y_k^*) \in \hat N_{\epsilon_k}((\bar x+t_ku_k, \bar y+t_kv_k), \operatorname{gph} G)$ and $\|y^*_{k|Z}\| \ge \alpha$.*

Here $y^*_{k|Z} \in Z^*$ denotes the restriction of y_k^* to Z. Consequently we have $\|y^*_{k|Z}\| = \sup_{z\in\mathcal{B}_Z}\langle y_k^*, z\rangle$.

We consider the case when Y can be represented as the topological direct sum of two subspaces Y_1, Y_2, i.e. $Y = Y_1 \oplus Y_2$. In what follows we denote by p_i, $i=1,2$ the projection from Y onto Y_i, i.e. $p_i \in \mathcal{L}(Y,Y_i)$, $p_i^2 = p_i$ and $y = p_1(y)+p_2(y)$, $\forall y$. For a multifunction $G : X \rightrightarrows Y$ we denote by G_i, $i=1,2$ the multifunction $G_i : X \rightrightarrows Y_i$ given by $G_i(x) = \{p_i(y) \mid y \in G(x)\}$

THEOREM 3.2. *Assume that G has closed graph , that Y admits a Fréchet smooth renorm and can be decomposed into the topological direct sum $Y_1\oplus Y_2$ of two subspaces Y_1, Y_2 such that the following assumptions are fulfilled for some neighborhoods U of $\bar x$, V of $p_1(\bar y)$ and positive constants κ, L:*

1. *For every x in U there is some $\hat x \in G_2^{-1}(p_2(\bar y))$ satisfying $\|x-\hat x\| \le \kappa \mathrm{d}(p_2(\bar y), G_2(x))$ and $G_1(x)\cap V \subset G_1(\hat x) + L\|x-\hat x\|\mathcal{B}_{Y_1}$.*
2. *$G(x) = G_1(x)+G_2(x)$, $\forall x \in G_2^{-1}(p_2(\bar y))\cap U$.*
3. *$(0,0) \notin \operatorname{Cr}_{Y_1} G(\bar x,\bar y)$.*

Then G is metrically subregular at $(\bar x,\bar y)$.

Proof. Let $\|\cdot\|$ denote the Fréchet smooth renorm of Y and set

$$|||y||| := \sqrt{\|p_1(y)\|^2 + (L\kappa+2)^2\|p_2(y)\|^2}.$$

Then $|||\cdot|||$ is again a Fréchet smooth renorm and, assuming that G is not subregular at $\bar x$ for $\bar y$, we can proceed as in the proof of the first part of theorem 2.2 with $\|y\|$ replaced by $|||y|||$. In order to prove the theorem we have to show that not only $(0,0) \in \operatorname{Cr} G(\bar x,\bar y)$ but also $(0,0) \in \operatorname{Cr}_{Y_1} G(\bar x,\bar y)$ follows. Since $\tilde x_k \to \bar x$, $\tilde y_k \to \bar y$ there is some index $\bar k$ such that $p_1(\bar y) + \|p_1(\tilde y_k-\bar y)\|B_{Y_1} \subset V$, $\tilde x_k + \kappa\|p_2(\tilde y_k-\bar y)\|B_X \subset U$ hold for all $k > \bar k$. Enlarging $\bar k$ if necessary we may also assume that

$$(1+\frac{1}{\sqrt{k}})(L\kappa+1) + \frac{\kappa}{\sqrt{k}} \le (1-\frac{1}{\sqrt{k}})(L\kappa+2),\ \forall k > \bar k. \tag{3.1}$$

For arbitrarily fixed $k > \bar k$ we have $\tilde x_k \in U$, $\tilde y_k \in G(\tilde x_k)$ and hence we can find some $\hat x_k \in G_2^{-1}(p_2(\bar y))$ such that $\|\hat x_k - \tilde x_k\| \le \kappa \mathrm{d}(p_2(\bar y), G_2(\tilde x_k)) \le \kappa\|p_2(\bar y) - p_2(\tilde y_k)\|$. Therefore $\hat x_k \in G_2^{-1}(p_2(\bar y)) \cap (\tilde x_k + \kappa\|p_2(\tilde y_k-\bar y)\|B_X) \subset G_2^{-1}(p_2(\bar y))\cap U$ implying $G(\hat x_k) = G_1(\hat x_k)+G_2(\hat x_k)$. Further we have $p_1(\tilde y_k) \in G_1(\tilde x_k)\cap V \subset G_1(\hat x_k) + L\|\tilde x_k - \hat x_k\|\mathcal{B}_{Y_1}$. Hence there is some $\hat y_k \in$

$G(\hat{x}_k)$ such that $p_2(\hat{y}_k) = p_2(\bar{y})$ and $\|p_1(\tilde{y}_k - \hat{y}_k)\| \le L\|\tilde{x}_k - \hat{x}_k\| \le L\kappa\|p_2(\bar{y} - \tilde{y}_k)\|$, implying $\varphi(\hat{x}_k, \hat{y}_k) = |||\hat{y}_k - \bar{y}||| = \|p_1(\hat{y}_k - \bar{y})\| \le L\kappa\|p_2(\bar{y} - \tilde{y}_k)\| + \|p_1(\tilde{y}_k - \bar{y})\|$. If there would hold $\|p_1(\tilde{y}_k - \bar{y})\| < \|p_2(\tilde{y}_k - \bar{y})\|$, then we could obtain

$$\begin{aligned} |||\tilde{y}_k - \bar{y}||| &= \varphi(\tilde{x}_k, \tilde{y}_k) \le \varphi(\hat{x}_k, \hat{y}_k) + \frac{\epsilon}{\alpha}\|(\hat{x}_k, \hat{y}_k) - (\tilde{x}_k, \tilde{y}_k)\| \\ &= \varphi(\hat{x}_k, \hat{y}_k) + \frac{1}{\sqrt{k}}(\|\tilde{x}_k - \hat{x}_k\| + |||\hat{y}_k - \tilde{y}_k|||) \\ &\le (1 + \frac{1}{\sqrt{k}})|||\hat{y}_k - \bar{y}||| + \frac{1}{\sqrt{k}}(\|\tilde{x}_k - \hat{x}_k\| + |||\tilde{y}_k - \bar{y}|||) \\ &< (1 + \frac{1}{\sqrt{k}})(L\kappa + 1)\|p_2(\bar{y} - \tilde{y}_k)\| + \frac{\kappa}{\sqrt{k}}\|p_2(\bar{y} - \tilde{y}_k)\| + \frac{1}{\sqrt{k}}|||\tilde{y}_k - \bar{y}||| \end{aligned}$$

and, after rearranging,

$$\left((1 + \frac{1}{\sqrt{k}})(L\kappa + 1) + \frac{\kappa}{\sqrt{k}}\right)\|p_2(\bar{y} - \tilde{y}_k)\| > (1 - \frac{1}{\sqrt{k}})|||\tilde{y}_k - \bar{y}||| \ge (1 - \frac{1}{\sqrt{k}})(L\kappa + 2)\|p_2(\bar{y} - \tilde{y}_k)\|$$

follows, a contradiction to (3.1). Hence $\|p_1(\tilde{y}_k - \bar{y})\| \ge \|p_2(\tilde{y}_k - \bar{y})\|$ holds and we can conclude $\|p_1(\tilde{y}_k - \bar{y})\| > 0$, since $\tilde{y}_k \ne \bar{y}$. Then the functional $\tilde{y}_k^*$, the Fréchet derivative of $|||\cdot|||$ at $\tilde{y}_k - \bar{y}$ has the representation

$$\langle \tilde{y}_k^*, h\rangle = \frac{\|p_1(\tilde{y}_k - \bar{y})\|\langle D\|p_1(\tilde{y}_k - \bar{y})\|, p_1(h)\rangle + (L\kappa + 2)^2\|p_2(\tilde{y}_k - \bar{y})\|\langle \xi_k^*, p_2(h)\rangle}{|||\tilde{y}_k - \bar{y}|||}, \forall h \in Y$$

where $\xi_k^* := D\|p_2(\tilde{y}_k - \bar{y})\|$ if $p_2(\tilde{y}_k - \bar{y}) \ne 0$ and $\xi_k^* \in S_{Y^*}$ can be chosen arbitrarily in case of $p_2(\tilde{y}_k - \bar{y}) = 0$. Hence

$$\langle \tilde{y}_k^*, p_1(\tilde{y}_k - \bar{y})\rangle = \frac{\|p_1(\tilde{y}_k - \bar{y})\|\langle D\|p_1(\tilde{y}_k - \bar{y})\|, p_1(\tilde{y}_k - \bar{y})\rangle}{|||\tilde{y}_k - \bar{y}|||} = \frac{\|p_1(\tilde{y}_k - \bar{y})\|^2}{|||\tilde{y}_k - \bar{y}|||} \ge \frac{\|p_1(\tilde{y}_k - \bar{y})\|}{\sqrt{1 + (L\kappa + 2)^2}}$$

showing $|||\tilde{y}^*_{k|Y_1}||| \ge (1 + (L\kappa + 2)^2)^{-1/2}$. Therefore $(0, 0) \in \operatorname{Cr}_{Y_1} G(\bar{x}, \bar{y})$ follows and the theorem is proved. □

Assumption 1) of theorem 3.2 requires that G_2 is metrically subregular at $(\bar{x}, p_2(\bar{y}))$. The condition on G_1 is for instance fulfilled if G_1 has the Aubin property near $(\bar{x}, \bar{y})$. Assumption 2) is e.g. fulfilled in the setting $Y = \hat{Y}_1 \times \hat{Y}_2$, $G(x) = \hat{G}_1(x) \times \hat{G}_2(x)$, where $G_i : X \rightrightarrows Y_i$, $i = 1, 2$ with $Y_1 = \hat{Y}_1 \times \{0_{\hat{Y}_2}\}$, $Y_2 = \{0_{\hat{Y}_1}\} \times \hat{Y}_2$, $G_1(x) = (\hat{G}_1(x), 0_{\hat{Y}_2})$, $G_2(x) = (0_{\hat{Y}_1}, G_2(x))$.

Now we present a characterization of the condition $(0, 0) \in \operatorname{Cr}_{Y_1} G(\bar{x}, \bar{y})$ in case of a smooth constraint system.

PROPOSITION 3.3. *Let $Z \subset Y$ denote a subspace. In case of a smooth constraint system we have $(0, 0) \in \operatorname{Cr}_Z G(\bar{x}, 0)$ if and only if G^{-1} is not locally upper Lipschitz at $(0, \bar{x})$ and there is a sequence $(\tilde{y}_k^*) \subset S_{Y^*}$ satisfying $\liminf_k \|\tilde{y}^*_{k|Z}\| > 0$, $\lim_k Dg(\bar{x})^*\tilde{y}_k^* = 0$, $\lim_k \langle \tilde{y}_k^*, g(\bar{x})\rangle - \sigma_C(\tilde{y}_k^*) = 0$*

Proof. If $(0, 0) \in \operatorname{Cr}_Z G(\bar{x}, 0)$ then also $(0, 0) \in \operatorname{Cr} G(\bar{x}, 0)$ holds and therefore G^{-1} is not locally upper Lipschitz at $(0, \bar{x})$ by proposition 2.6. Next consider a sequence $(t_k, \epsilon_k, v_k, x_k^*, u_k, y_k^*)$ together with some constant α according to the definition 3.1 and set $x_k := \bar{x} + t_k u_k$, $y_k := 0 + t_k v_k$, $c_k = g(x_k) - y_k$. For each $c \in C$ and each $\epsilon_k' > \epsilon_k$ we easily deduce from $(-x_k^*, y_k^*) \in \hat{N}_{\epsilon_k}((x_k, y_k), \operatorname{gph} G)$ together with some $\lambda \in (0, 1]$ such that $c_k + \lambda(c - c_k) \in C$ is sufficiently close to c_k, that

$$\begin{aligned} &\langle -x_k^*, x_k - x_k\rangle + \langle y_k^*, g(x_k) - (c_k + \lambda(c - c_k)) - y_k\rangle \\ &= \langle -y_k^*, \lambda(c - c_k))\rangle \le \epsilon_k'(\|x_k - x_k\| + \|g(x_k) - (c_k + \lambda(c - c_k)) - y_k\|) = \epsilon_k'\|\lambda(c - c_k)\|. \end{aligned}$$

Hence $-y_k^* \in \hat{N}_{\epsilon_k}(c_k, C)$. Therefore $\operatorname{d}(-y_k^*, N_C(c_k)) \le \epsilon_k$ and without loss of generality we can assume $\tilde{y}_k^* := -y_k^* \in N_C(c_k)$ implying $\langle \tilde{y}_k^*, g(x_k) - y_k\rangle - \sigma_C(\tilde{y}_k^*) = 0$. Since $g(\bar{x}) \in C$ we obtain

$0 \geq \langle \tilde{y}_k^*, g(\bar{x})\rangle - \sigma_C(\tilde{y}_k^*) = \langle \tilde{y}_k^*, c_k\rangle - \sigma_C(\tilde{y}_k^*) + \langle \tilde{y}_k^*, g(\bar{x}) - c_k\rangle \geq -\|g(\bar{x}) - c_k\|$ and $\lim_k \langle \tilde{y}_k^*, g(\bar{x})\rangle - \sigma_C(\tilde{y}_k^*) = 0$ follows. The condition $\liminf_k \|\tilde{y}^*_{k|Z}\| > 0$ follows immediately from $\|y^*_{k|Z}\| \geq \alpha > 0$ and there remains to show $\lim_k Dg(\bar{x})^* \tilde{y}_k^* = 0$. Using $(-x_k^*, y_k^*) \in \hat{N}_{\epsilon_k}((x_k, y_k), \operatorname{gph} G)$ we obtain for each $x \in x_k + \rho_k \mathcal{B}_X$, where $\rho_k > 0$ is chosen sufficiently small, that

$$\begin{aligned}\langle y_k^*, g(x) - g(x_k)\rangle &= \langle y_k^*, (g(x) - c_k) - y_k\rangle \\ &\leq \langle x_k^*, x - x_k\rangle + (\epsilon_k + \frac{1}{k})(\|x - x_k\| + \|g(x) - g(x_k)\|) \\ &\leq (\epsilon_k + \frac{1}{k} + \|x_k^*\|)(1 + \|Dg(\bar{x})\|)\|x - x_k\|\end{aligned}$$

and consequently

$$\begin{aligned}\langle Dg(\bar{x})^* y_k^*, x - x_k\rangle &= \langle y_k^*, g(x) - g(x_k)\rangle - \langle y_k^*, g(x) - g(x_k) - Dg(\bar{x})(x - x_k)\rangle \\ &\leq (\epsilon_k + \frac{1}{k} + \|x_k^*\|)\|x - x_k\|(1 + \|Dg(\bar{x})\|) + \|g(x) - g(x_k) - Dg(\bar{x})(x - x_k)\|.\end{aligned}$$

From this we can easily deduce that $\|Dg(\bar{x})^* \tilde{y}_k^*\| \leq (\epsilon_k + \frac{1}{k} + \|x_k^*\|)(1 + \|Dg(\bar{x})\|) + \eta_k$, where $\eta_k = \sup\{\frac{\|g(x) - g(x_k) - Dg(\bar{x})(x - x_k)\|}{\|x - x_k\|} \mid x \in x_k + \rho_k \mathcal{B}_X\} \to 0$ due to strict differentiability of g, and the required condition $\lim_k Dg(\bar{x})^* \tilde{y}_k^* = 0$ follows.

On the other hand, if G^{-1} is not locally upper Lipschitz at $(\bar{y}, \bar{x})$ and there exists a sequence $\tilde{y}_k^*$ with the specified properties, then we can argue as in the proof of proposition 2.7 to show $(0,0) \in \operatorname{Cr} G(\bar{x}, \bar{y})$. But from the condition $\liminf_k \|\tilde{y}^*_{k|Z}\| > 0$ we can also deduce $(0,0) \in \operatorname{Cr}_Z G(\bar{x}, \bar{y})$. □

We now take a closer look at the characterizations of proposition 3.3

PROPOSITION 3.4. *In case of a smooth constraint system we have:*

1. G^{-1} *is locally upper Lipschitz at* $(0, \bar{x})$ *if and only if there is no sequence* $(u_k) \subset \mathcal{S}_X$ *satisfying* $\lim_k \mathrm{d}(Dg(\bar{x})u_k, T_C(g(\bar{x}))) = 0$.

If the subspace $Dg(\bar{x})X + \operatorname{lin}(C - g(\bar{x}))$ *is closed in* Y *and the subspace* $Dg(\bar{x})X \cap \operatorname{lin}(C - g(\bar{x}))$ *is finite dimensional,* G^{-1} *is locally upper Lipschitz at* $(0, \bar{x})$ *if and only if there in no nonzero* u *satisfying* $Dg(\bar{x})u \in T_C(g(\bar{x}))$.

2. *Let* $Z \subset Y$ *denote a subspace and assume that the set* $(Dg(\bar{x})\mathcal{B}_X - (C - g(\bar{x}))) \cap Z$ *has nonempty interior in* Z*. Then there is a sequence* $(\tilde{y}_k^*) \subset S_{Y^*}$ *satisfying* $\liminf_k \|\tilde{y}^*_{k|Z}\| > 0$, $\lim_k Dg(\bar{x})^* \tilde{y}_k^* = 0$, $\lim_k \langle \tilde{y}_k^*, g(\bar{x})\rangle - \sigma_C(\tilde{y}_k^*) = 0$ *if and only if there is some element* $y^* \in Y^*$ *satisfying* $\|y^*_{|Z}\| > 0$, $Dg(\bar{x})^* y^* = 0$, $\langle y^*, g(\bar{x})\rangle - \sigma_C(\tilde{y}^*) = 0$

Proof.

1. By the definition G^{-1} is not locally upper Lipschitz at $(0, \bar{x})$ if and only if there is a sequence $(x_k, y_k) \in \operatorname{gph} G$ converging to $(\bar{x}, 0)$ such that $t_k := \|x_k - \bar{x}\| > M_k \|y_k\|$ for some sequence $M_k \to \infty$. Denoting $u_k = t_k^{-1}(x_k - \bar{x}) \in \mathcal{S}_X$, $c_k = g(x_k) - y_k$ we obtain

$$\frac{1}{M_k} > t_k^{-1}\|g(x_k) - c_k\| = \|Dg(\bar{x})u_k - \frac{o(t_k)}{t_k} - \frac{c_k - g(\bar{x})}{t_k}\|$$

and, since $t_k^{-1}(c_k - g(\bar{x})) \in T_C(g(\bar{x}))$, we have $\lim_k \mathrm{d}(Dg(\bar{x})u_k, T_C(g(\bar{x}))) = 0$. On the other hand, if there is a sequence $(u_k) \subset \mathcal{S}_X$ satisfying $\lim_k \mathrm{d}(Dg(\bar{x})u_k, T_C(g(\bar{x}))) = 0$, we can choose sequences $(t_k) \downarrow 0$ and $(c_k) \subset C$ with $\lim_k \tau_k = 0$, where $\tau_k := \|Dg(\bar{x})u_k - t_k^{-1}(c_k - g(\bar{x}))\|$ By taking $x_k := \bar{x} + t_k u_k$, $y_k = g(x_k) - c_k$ we obtain

$$\|y_k\| = t_k \tau_k + o(t_k) = \|x_k - \bar{x}\|(\tau_k + \frac{o(t_k)}{t_k})$$

showing that G^{-1} is not locally upper Lipschitz. To show the second assertion, let $(u_k) \subset \mathcal{S}_X$ and $(h_k) \subset T_C(g(\bar{x})) \subset \operatorname{lin}(C - g(\bar{x}))$ denote sequences fulfilling $w_k := Dg(\bar{x})u_k - h_k \to 0$. Obviously we have $w_k \in Dg(\bar{x})X + \operatorname{lin}(C - g(\bar{x}))$. By the Generalized Open Mapping Theorem [18, Theorem 1] the multifunction $u \rightrightarrows Dg(\bar{x})u + \operatorname{lin}(C - g(\bar{x}))$ acting between X and the Banach space $Dg(\bar{x})X + \operatorname{lin}(C - g(\bar{x}))$ is metrically regular near $(0,0)$ with some modulus κ and hence for each k sufficiently large there is some $\tilde{u}_k \in X$ such that $-w_k \in Dg(\bar{x})\tilde{u}_k + \mathrm{lin}\,(C - g(\bar{x}))$ and $\|\tilde{u}_k\| \leq \kappa\|-w_k\| \to 0$. Therefore $0 \in Dg(\bar{x})(u_k+\tilde{u}_k)+\mathrm{lin}\,(C-g(\bar{x}))$ implying that $Dg(\bar{x})(u_k+\tilde{u}_k)$ belongs to the finite dimensional space $Dg(\bar{x})X \cap \mathrm{lin}\,(C-g(\bar{x}))$. Hence we can find some bounded sequence (v_k) belonging to some finite dimensional subspace of X such that $Dg(\bar{x})(u_k+\tilde{u}_k-v_k) = 0$. By passing to a subsequence if necessary we can assume that (v_k) converges to some $\bar{v}$. If $\bar{v} = 0$ then we can take $u = u_k+\tilde{u}_k-v_k \neq 0$ for some k sufficiently large showing $Dg(\bar{x})u = 0 \in T_C(g(\bar{x}))$. Otherwise we can take $u = \bar{v}$ to obtain

$$Dg(\bar{x})u = \lim_k Dg(\bar{x})v_k = \lim_k Dg(\bar{x})(u_k + \tilde{u}_k) = \lim_k h_k \in T_C(g(\bar{x})).$$

2. Let $(\tilde{y}_k^*) \subset \mathcal{S}_{Y^*}$ denote a sequence fulfilling $\liminf_k \|\tilde{y}^*_{k|_Z}\| =: \alpha > 0$, $\lim_k Dg(\bar{x})^*\tilde{y}_k^* = 0$, $\lim_k \langle \tilde{y}_k^*, g(\bar{x})\rangle - \sigma_C(\tilde{y}_k^*) = 0$. By the Alaoglu-Bourbaki Theorem the sequence $(\tilde{y}_k^*)$ has a weak-$*$ limit point y^* fulfilling $Dg(\bar{x})^*y^* = 0$ and, due to the weak-$*$ lower semicontinuity of the support function $\sigma_C(\cdot)$, $\langle y^*, g(\bar{x})\rangle - \sigma_C(y^*) \geq 0$. Since $g(\bar{x}) \in C$, the condition $\langle y^*, g(\bar{x})\rangle - \sigma_C(y^*) = 0$ follows. Now choose $\tilde{z} \in Z$ and some radius $\rho > 0$ such that $\tilde{z} + \rho\mathcal{B}_Z \subset Dg(\bar{x})\mathcal{B}_X - (C - (g(\bar{x})))$. Then

$$\begin{aligned}\inf_{z\in\tilde{z}+\rho\mathcal{B}_Z} \langle \tilde{y}_k^*, z\rangle = \langle \tilde{y}_k^*, \tilde{z}\rangle - \rho\|\tilde{y}^*_{k|_Z}\| &\geq \inf_{\substack{x\in\mathcal{B}_X\\ c\in C}} \langle \tilde{y}_k^*, Dg(\bar{x})x - (c - g(\bar{x}))\rangle \\ &= -\|Dg(\bar{x})^*\tilde{y}_k^*\| + \langle \tilde{y}_k^*, g(\bar{x})\rangle - \sigma_C(\tilde{y}_k^*) \to 0\end{aligned}$$

and therefore $\langle y^*, \tilde{z}\rangle \geq \liminf_k \langle \tilde{y}_k^*, \tilde{z}\rangle \geq \rho\alpha > 0$ showing $\|y^*_{|_Z}\| > 0$.
□

Remark 3.5.

1. The subspace $Dg(\bar{x})X + \mathrm{lin}\,(C - g(\bar{x}))$ is closed in Y and the subspace $Dg(\bar{x})X \cap \mathrm{lin}\,(C - g(\bar{x}))$ is finite dimensional, if any of the following assumptions is fulfilled:
 a) X is finite dimensional,
 b) Y is finite dimensional,
 c) $Dg(\bar{x})$ has closed range and $\mathrm{lin}\,C$ is finite dimensional.
2. If $Z \subset Y$ is a closed subspace then, as a consequence of the Generalized Open Mapping Theorem [18, Theorem 1], the set $(Dg(\bar{x})\mathcal{B}_X - (C - g(\bar{x}))) \cap Z$ has nonempty interior in Z if and only if this holds for the larger set $(Dg(\bar{x})X - (C - g(\bar{x}))) \cap Z$.
3. Note that the condition $\langle y^*, g(\bar{x})\rangle - \sigma_C(y^*) = 0$ is equivalent with $y^* \in N_C(g(\bar{x}))$.
4. Of course the second part of proposition 3.4 also holds for the special case $Z = Y$. Let us note that the existence of some sequence $(\tilde{y}_k^*) \subset S_{Y^*}$ satisfying $\lim_k Dg(\bar{x})^*\tilde{y}_k^* = 0$, $\lim_k \langle \tilde{y}_k^*, g(\bar{x})\rangle - \sigma_C(\tilde{y}_k^*) = 0$ is equivalent with the condition

$$\sup_{y^*\in\mathcal{S}_{Y^*}} \langle y^*, g(\bar{x})\rangle - \sigma_C(y^*) - \|Dg(\bar{x})^*y^*\| = 0$$

and consequently also equivalent with the condition $0 \notin \mathrm{int}\, Dg(\bar{x})B_X - (C - (g(\bar{x}))$ (see [4]), which tells us that Robinson's constraint qualification $0 \in \mathrm{int}\, Dg(\bar{x})X - (C - g(\bar{x}))$ does not hold and therefore G is not metrically regular near $(\bar{x}, 0)$, cf. [1]

To illustrate theorem 3.2 and propositions 3.3, 3.4 we consider the following example:

Example 3.6. Let $G : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$, $G(x) = g(x) - C$ be given by

$$g(x_1, x_2, x_3) = \begin{pmatrix} x_1 - x_2 + x_3 + x_1^3 \\ x_2 - x_3 - x_1^3 \\ -x_1 \end{pmatrix}, \quad C = \{0\} \times \mathbb{R}^2_-$$

and let $\bar{x} = (0, 0, 0)$, $\bar{y} = (0, 0, 0)$. Then

$$Dg(\bar{x}) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in T_C(g(\bar{x}))$$

and hence G^{-1} is not upper Lipschitz due to proposition 3.4. Further we have

$$\Lambda := \{y^* \in Y^* \mid \begin{matrix} \langle y^*, g(\bar{x})\rangle - \sigma_C(y^*) = 0 \\ Dg(\bar{x})^* y^* = 0 \end{matrix}\} = \{\begin{pmatrix} y_1^* \\ y_2^* \\ y_3^* \end{pmatrix} \in \mathbb{R} \times \mathbb{R}^2_+ \mid \begin{matrix} y_1^* - y_3^* & = & 0 \\ -y_1^* + y_2^* & = & 0 \\ y_1^* - y_2^* & = & 0 \end{matrix}\}$$

$$= \mathbb{R}_+ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \neq \{0\}$$

and from remark 3.5 we conclude that G is not metrically regular at $(\bar{x}, 0)$. Hence from proposition 2.7 it follows that $(0,0) \in \operatorname{Cr} G(\bar{x}, 0)$.

We will now invoke theorem 3.2 to show that G is metrically subregular at $(\bar{x}, 0)$. Consider the decomposition $Y = Y_1 \oplus Y_2$, where

$$Y_1 := \mathbb{R}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad Y_2 := \{\begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{R}\}.$$

Then for all $y^* \in \Lambda$ we have $y^*_{|Y_1} = 0$ and from propositions 3.4, 3.3 we conclude $(0,0) \notin \operatorname{Cr}_{Y_1} G(\bar{x}, 0)$. Further we have

$$p_1(y_1, y_2, y_3) = \begin{pmatrix} y_1 \\ -y_1 \\ 0 \end{pmatrix}, \quad p_2(y_1, y_2, y_3) = \begin{pmatrix} 0 \\ y_1 + y_2 \\ y_3 \end{pmatrix}$$

and

$$G_1(x_1, x_2, x_3) = \{\begin{pmatrix} x_1 - x_2 + x_3 + x_1^3 \\ -x_1 + x_2 - x_3 - x_1^3 \\ 0 \end{pmatrix}\}, \quad G_2(x_1, x_2, x_3) = \begin{pmatrix} 0 \\ x_1 \\ -x_1 \end{pmatrix} - \{0\} \times \mathbb{R}^2_-.$$

G_2 is a polyhedral multifunction and therefore metrically subregular at $(\bar{x}, p_2(0))$. Further G_1 is single-valued and Lipschitz near $\bar{x}$ and therefore Assumption 1) of theorem 3.2 is fulfilled. Since also $G = G_1 + G_2$ holds, we can apply Theorem 3.2 to show that G is metrically subregular at $(\bar{x}, 0)$.

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