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GENERALIZED PENALTY METHODS FOR A CLASS OF CONVEX OPTIMIZATION PROBLEMS WITH POINTWISE INEQUALITY CONSTRAINTS

HELMUT GFRERER*

Abstract. We consider infinite-dimensional convex minimization problems with pointwise inequality constraints. We do not impose a constraint qualification condition, so that first-order necessary conditions are not available. To handle such problems we propose a method which combines advantages both from well-known penalty and barrier methods. For solving the resulting subproblems we develop a Newton-type algorithm in function space and prove its global and locally superlinear convergence. We demonstrate our results by means of an minimization problem subject to a elliptic partial differential equation and pointwise control, zero- and first-order state constraints. Numerical results are presented.

Key words. Pointwise constraints, penalty and barrier methods, constraint qualification, Newton method.

AMS subject classifications. 49M30, 65K05, 90C48

1. Introduction. Throughout this paper we focus on convex problems of the form

$$(P) \qquad \min_{z\in Z} J(z)$$

subject to

$$Ez = 0$$
$$g_i(z) \le \varphi_i \quad \mu_i\text{-a.e. in } \Omega_i,\ i = 1,\dots,m$$

where the cost functional $J : Z \to \mathbb{R}$ is defined on a Hilbert space Z and the operator $E \in \mathcal{L}(Z, V)$ is a bounded linear operator from Z into another Banach space V. Further we have finitely many pointwise inequality constraints defined by mappings $g_i : Z \to L^{r_i}(\Omega_i)$ and bounds $\varphi_i \in L^{r_i}(\Omega_i)$, $i = 1\dots,m$, where for each i, we have $r_i \ge 1$ and Ω_i denotes a finite measure space $(\Omega_i, \mathcal{A}_i, \mu_i)$ given by the set Ω_i, a σ-algebra $\mathcal{A}_i \subset \mathcal{P}(\Omega_i)$ of subsets of Ω_i and a finite measure μ_i on $\mathcal{A}_i$. Here we use the usual convention to denote a measure space by the same symbol as the underlying set. In what follows, we denote by $Z_E = \operatorname{Ker} E$ the kernel of E and consider it as a subspace of Z.

Throughout the paper we require the following hypothesis on the problem (P)

(P1) There exists a feasible point z_f for the constraints in (P)

(P2) The cost functional J is convex and lower semi-continuous on Z_E.

(P3) There exists some positive real α such that for all $z_1, z_2 \in Z_E$ we have

$$J(z_2) \ge J(z_1) + J'(z_1, z_2 - z_1) + \frac{\alpha}{2}\|z_2 - z_1\|^2$$

(P4) For each $i = 1,\dots,m$, and for each closed convex set $U \subset L^{r_i}(\Omega_i)$, the set $\{z \in Z_E : g_i(z) \in U\}$ is closed. Further, for all $z_1, z_2 \in Z_E$ and all $t \in [0,1]$ we have

$$tg_i(z_1)(x) + (1-t)g_i(z_2)(x) \ge g_i(tz_1 + (1-t)z_2)(x) \text{ for almost all } x \in \Omega_i.$$

Here, $J'(z_1, z_2 - z_1)$ denotes the directional derivative at z_1 in direction $z_2 - z_1$. For a function $f : X \to \bar{\mathbb{R}}$, acting on a topological linear space X, we say that f is directionally differentiable at x in direction $h \in X$, if the limit

$$f'(x,h) = \lim_{t\downarrow 0} \frac{f(x+th) - f(x)}{t}$$

exists. Recall that the directional derivative $f'(x,h)$ exists if $t \to f(x+th)$ is convex and finite on some interval $(-\epsilon, \epsilon)$.

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Assumption (P4) guarantees that the feasible region of the problem (P) is convex.

The problem (P) is a rather general problem. Two-sided inequality constraints of the form $\underline{\varphi}_i \le g_i(z) \le \overline{\varphi}_i$ can be simply written as two inequalities $g_i(z) \le \overline{\varphi}_i$ and $-g_i(z) \le -\underline{\varphi}_i$, where of course our convexity assumptions require that g_i is linear. The case of scalar convex inequalities can be treated by choosing a finite set Ω_i. Further note that problems with inhomogeneous constraints $Ez = f$ can be equivalently transformed into (P) by the substitution $\hat{z} = z - z_0$, where z_0 fulfills $Ez_0 = f$.

In recent years a number of research efforts focused on efficient methods for solving constrained variational problems in function space, see e.g. the references in the monograph [9]. Many of these problems are of the form of the problem (P), as an example we concentrate in this paper on the following control problem:

Example 1 The optimal control problem

$$\begin{aligned} \min_{z=(y,u)} J(z) &= \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|u - u_d\|^2_{L^2(\Omega)} \\ \text{subject to } -\triangle y &= u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega \\ u &\le \varphi_u \quad \text{a.e. in } \Omega \\ y &\le \varphi_y \quad \text{a.e. in } \Omega \\ |\nabla y|_2 &\le \varphi_g \quad \text{a.e. in } \Omega \end{aligned}$$

where $\beta > 0$, Ω is a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary, $y_d, u_d, \varphi_u, \varphi_y, \varphi_g \in L^2(\Omega)$ and $|\cdot|_p$ denotes the ℓ^p-norm in $\mathbb{R}^d$, is a special case of (P) with $m = 3$, $\Omega_i = \Omega$, $i = 1, \ldots, 3$, μ_i equal to the Lebesgue measure on Ω,

$$\begin{aligned} &Z = H^1_0(\Omega) \times L^2(\Omega),\ V = H^{-1}(\Omega),\ \langle E(y,u), v\rangle = \int_\Omega \nabla y \cdot \nabla v - \int_\Omega uv, \\ &g_1(y,u) = u,\ \varphi_1 = \varphi_u, \\ &g_2(y,u) = y,\ \varphi_2 = \varphi_y, \\ &g_3(y,u) = |\nabla y|_2,\ \varphi_3 = \varphi_g. \end{aligned}$$

Further the assumptions (P2)–(P4) are satisfied.

The Assumptions (P1)–(P4) guarantee the existence of a unique solution $\bar{z}$ for the problem (P). Indeed, from (P2) and the Corollaries 2.4, 2.5 in [3, Chapter 1] we conclude that J is Lipschitz near z_f with some modulus C and consequently we obtain from (P3), that $J(z) \ge J(z_f) - C\|z - z_f\|_Z + \frac{\alpha}{2}\|z - z_f\|^2_Z$, $\forall z \in Z_E$. Noting that by (P3) also strict convexity of $\tilde{J}$ follows and that by (P4) the feasible region for the problem (P) is a closed convex subset of the Hilbert space Z_E, the existence of a unique solution follows from Proposition 1.2 in [3, Chapter 2].

However, these arguments for showing existence are non-constructive and do not lead to an efficient numerical algorithm. Further note that the usual first order optimality conditions do not hold for the general problem (P) unless we impose an additionally constraint qualification condition. For the control problem of Example 1 it is well known that no multipliers exist in this setting, for a discussion on a related problem see e.g. [6]. To derive first order optimality conditions one can sometimes use a different setting much more complicated than that where we can show the unique existence of a solution. Usually this approach leads to measure valued multipliers and this complicates the numerical treatment considerably.

So we choose a penalization approach which avoids the use of multipliers. We consider a very general class of penalty methods, which comprises both barrier and penalty methods. In the case of PDE-constrained optimization barrier methods have been successfully applied to some special problems, see e.g.[12], [13], [15], [16], [17], [18]. Our approach is more related to the penalty method used in [6], [7], [8], [10], where the penalization is considered as a regularization for handling measure-valued multipliers.

The remainder of the paper is organized as follows. In sections 2 and 3 we introduce the generalized penalty method for the problem (P) and prove convergence together with some error estimate. In section 4 we state some duality results. We will see that whenever a standard constraint qualification for convex programming in a possibly different setting is fulfilled which guarantees the existence of multipliers, then we can also retrieve multipliers from the penalty method. In section 5 we state a Newton-type algorithm for solving the subproblems and show global and superlinear convergence properties. Section 6 ends the paper with a report on some numerical experiments carried out for the problem from Example 1.

2. Generalized penalty methods for solving (P). It is clear that $\bar{z}$ is also the solution of the problem

$$\min_z J(z) + \Psi(z) := J(z) + \sum_{i=1}^m \int_{\Omega_i} \iota_{(-\infty,0]}(g_i(z)(x) - \varphi_i(x))\, d\mu_i(x) \quad \text{s.t.} \quad Ez = 0,$$

where

$$\iota_{(-\infty,0]}(t) = \begin{cases} 0 & \text{if } t \le 0 \\ +\infty & \text{if } t > 0 \end{cases}$$

denotes the indicator function of the set of non-positive real numbers.

We use the notion of a generalized penalty method for choosing a triple $(Q,\ \bar{q},\ (\psi_{i,q})_{i=1,\dots,m,q\in Q})$, where the parameter set Q is some topological space and $\bar{q} \in Q$ is a target parameter, which is assumed to have a countable base of neighborhoods. For each $q \in \dot{Q} := Q \setminus \{\bar{q}\}$ and each $i \in \{1,\dots,m\}$ we are given some function $\psi_{i,q} : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, such that $\psi_{i,q}$ approaches $\iota_{(-\infty,0]}$ for $q \to \bar{q}$. The goal now is to approximate the solution $\bar{z}$ by solutions $\bar{z}_q$ of the problem

$$(P_q) \qquad \min_z J_q(z) := J(z) + \Psi_q(z) := J(z) + \sum_{i=1}^m \int_{\Omega_i} \psi_{i,q_i}(g_i(z)(x) - \varphi_i(x))\, d\mu_i(x)$$
$$\text{s.t.} \qquad Ez = 0.$$

In what follows we denote by $\mathcal{Q}$ the system of neighborhoods of $\bar{q}$ and by $\mathcal{Z}(z)$ the system of weak neighborhoods of z in Z_E.

THEOREM 2.1. *Assume that for each $q \in \dot{Q}$ the function $\Psi_q : Z \to \mathbb{R} \cup \{\infty\}$ is well defined, convex, l.s.c. on Z_E and $\Psi_q(z_f) < \infty$. Further assume that for each $i \in \{1,\dots,m\}$ there is some linear functional $\eta_i^* \in Z_E^*$ and families of real numbers $(a_{i,q})_{q\in\dot{Q}}$, $(b_{i,q})_{q\in\dot{Q}}$ with $\lim_{q\to\bar{q}} a_{i,q} = \lim_{q\to\bar{q}} b_{i,q} = 0$ such that $\Psi_q(z) \ge \sum_{i=1}^m (b_{i,q}\langle \eta_i^*, z\rangle + a_{i,q})$, $\forall z \in Z_E$. Then for each $q \in \dot{Q}$ the problem (P_q) has a unique solution denoted by $\bar{z}_q$. If in addition*

1. *for each $z \in Z_E$ not feasible for the problem (P) and for each real R there are some neighborhoods $U_z \in \mathcal{Z}(z)$ and $U_q \in \mathcal{Q}$ such that*

$$\inf_{q\in U_q} \inf_{z'\in U_z} \Psi_q(z') \ge R, \tag{2.1}$$

and

2. *for each $\hat{z} \in Z_E$ feasible for the problem (P) there is some family $(\hat{z}_q)_{q\in\dot{Q}} \subset Z_E$ with $\lim_{q\to\bar{q}} \hat{z}_q = \hat{z}$ such that $\limsup_{q\to\bar{q}} \Psi_q(\hat{z}_q) \le 0$,*

then $\lim_{q\to\bar{q}} \bar{z}_q = \bar{z}$.

Proof. By $(P2)$ the function $z \to J(z) + \Psi_q(z)$ is convex and l.s.c. on Z_E. By using Corollaries 2.4, 2.5 in [3, Chapter 1] we conclude that J is Lipschitz on Z_E near 0 with some modulus C and consequently we obtain from (P3), that $J(z) \ge J(0) - C\|z\|_{Z_E} + \frac{\alpha}{2}\|z\|^2_{Z_E}$, $\forall z \in Z_E$. Hence $J(z) + \Psi_q(z) \ge J(0) - (C + \sum_{i=1}^m |b_{i,q}|\|\eta_i^*\|_{Z_E^*})\|z\|_{Z_E} - \sum_{i=1}^m |a_{i,q}| + \frac{\alpha}{2}\|z\|^2_{Z_E}$ and thus $J(z) + \Psi_q(z)$ is coercive on Z_E. The existence of a unique solution $\bar{z}_q$ follows now from Proposition 1.2 in [3, Chapter 2]. Taking a family $(z_{f,q})_{q\in\dot{Q}} \subset Z_E$ converging to z_f such that $\limsup_{q\to\bar{q}} \Psi_q(z_{f,q}) \le 0$ we obtain with $\tilde{C}_q = C + \sum_{i=1}^m |b_{i,q}|\|\eta_i^*\|_{Z_E^*}$

$$J(z_{f,q}) + \Psi_q(z_{f,q}) \ge J(\bar{z}_q) + \Psi_q(\bar{z}_q) \ge J(0) - \tilde{C}_q\|\bar{z}_q\|_{Z_E} - \sum_{i=1}^m |a_{i,q}| + \frac{\alpha}{2}\|\bar{z}_q\|^2_{Z_E}.$$

We can find some neighborhood N of $\bar q$ such that $J(z_{f,q})+\Psi_q(z_{f,q})+\sum_{i=1}^m |a_{i,q}|\le J(z_f)+1$ and $\tilde C_q\le C+1$ holds for every $q\in N$. Hence the set $\{\bar z_q\,:\,q\in N\}$ is bounded.

To show $\lim_{q\to\bar q}\bar z_q=\bar z$ it is sufficient to prove $\lim_n \bar z_{q^n}=\bar z$ for every sequence $(q^n)\to q$. Let (q^n) be an arbitrary sequence converging to $\bar q$. Since J is convex and l.s.c, on Z_E it is also weakly l.s.c. Thus, for every $z\in Z_E$ and every $\epsilon>0$ we can find some neighborhood $U_z\in\mathcal Z(x)$ such that $J(z')\ge J(z)-\epsilon$ and $|\langle\eta_i^*,z'-z\rangle|\le 1$, $i=1,\dots,m$ holds for all $z'\in U_z$. Then we can also find some neighborhood $U_q\in\mathcal Q$ such that $\Psi_q(z')\ge-\epsilon$, $\forall z'\in U_z$, $\forall q\in U_q$. Hence $\inf_{q\in U_q}\inf_{z'\in U_z}J(z')+\Psi_q(z')\ge J(z)-2\epsilon$ holds and therefore for each z feasible for the problem (P) we have

$$\sup_{U_z\in\mathcal Z(z)}\liminf_n\inf_{z'\in U_z}J(z')+\Psi_{q^n}(z')\ge J(z)+\Psi(z).$$

Due to (2.1) this condition also holds when z is not feasible for the problem (P). If z is feasible for the problem (P), by our assumptions we can find a sequence $(z_n)\subset Z_E$ converging strongly to z with $\limsup_n\Psi_{q^n}(z_n)\le 0$. Since any weak neighborhood $U_z\in\mathcal Z(z)$ is also a neighborhood in the strong topology on Z_E and J is continuous on Z_E, we obtain

$$\sup_{U_z\in\mathcal Z(z)}\limsup_n\inf_{z'\in U_z}J(z')+\Psi_{q^n}(z')\le J(z)+\Psi(z).$$

This condition is obviously true for non feasible z. We conclude that the sequence $J(z)+\Psi_{q^n}$ weakly epi-converges to $J+\Psi$ and hence any weakly convergent subsequence of $(\bar z_{q^n})$ converges to $\bar z$ by [1, Theorem 1.10]. By the Eberlein-Shmulyan Theorem (see, e.g. [19]), every bounded sequence in the Hilbert space Z_E has a weakly convergent subsequence. Since the sequence $(\bar z_{q^n})$ is bounded it follows that $(\bar z_{q^n})$ converges weakly to $\bar z$. To show strong convergence, let (z_n) denote a sequence converging to $\bar z$ with $\limsup_n\Psi_{q^n}(z_n)\le 0$. Since J is Lipschitz near $\bar z$ on Z_E we can choose some $\bar z^*\in\partial J(\bar z)\subset Z_E^*$ and together with $J'(\bar z,z-\bar z)\ge\langle\bar z^*,z-\bar z\rangle$, $z\in Z_E$ we obtain from (P3)

$$\begin{aligned}J(\bar z)&\ge\limsup_n(J(z_n)+\Psi_{q^n}(z_n))\ge\limsup_n(J(\bar z_{q^n})+\Psi_{q^n}(\bar z_{q^n}))\\&\ge\limsup_n(J(\bar z)+\langle\bar z^*,\bar z_{q^n}-\bar z\rangle+\frac{\alpha}{2}\|\bar z_{q^n}-\bar z\|^2_{Z_E}+\sum_{i=1}^m(a_{i,q^n}+b_{i,q^n}\langle\eta_i^*,\bar z_{q^n}\rangle))\\&\ge J(\bar z)+\limsup_n\frac{\alpha}{2}\|\bar z_{q^n}-\bar z\|^2_{Z_E},\end{aligned}$$

implying $\limsup_n\|\bar z_{q^n}-\bar z\|_{Z_E}=0$. □

In what follows we make the following basic assumptions on the functions $\psi_{i,q}$ for every $i\in\{1,\dots,m\}$ and every $q\in\dot Q$:

(Q1) $\psi_{i,q}$ is a convex, l.s.c. and increasing function with $(-\infty,0)\subset\operatorname{dom}\psi_{i,q}$.

(Q2) $\lim_{q\to\bar q}\psi_{i,q}(t)=\iota_{(-\infty,0]}(t)$, $\forall t\ne 0$.

We present some examples for generalized penalty functions for the case $m=1$, where we drop the index i:

1. The *quadratic penalty function* is given by $Q=[0,\infty]$, $\bar q=\infty$ and $\psi_\kappa(t)=\kappa\max\{0,t\}^2$ for every $\kappa\in\dot Q$

2. In case of the *logarithmic barrier function* we have $Q=\mathbb R_+$, $\bar q=0$,

$$\psi_\kappa(t)=\begin{cases}-\kappa\ln(-t)&\text{if } t<0,\\ \infty&\text{if } t\ge 0.\end{cases}$$

3. The *combined logarithmic-quadratic penalty function* is given by the setting $Q=\{(\kappa,\epsilon)\in\mathbb R^2_+\,:\,\kappa\ge\alpha\epsilon^{3/2}>0,\kappa^{1/2}\ln\epsilon>-\beta\}$, where α,β are some positive constants, $\bar q=(0,0)$ and

$$\psi_{(\kappa,\epsilon)}(t)=\begin{cases}-\kappa\ln(-t)&\text{if } t\le-\epsilon,\\ \kappa(-\ln(\epsilon)+\frac{t+\epsilon}{\epsilon}+\frac{(t+\epsilon)^2}{2\epsilon^2}&\text{if } t>-\epsilon,\end{cases}\tag{2.2}$$

for every $(\kappa, \epsilon) \in \dot{Q}$. Note that for each fixed $(\kappa, \epsilon) \in \dot{Q}$ we have $\psi_{(\kappa,\epsilon)} \in C^2(\mathbb{R})$, $|D^2\psi_{(\kappa,\epsilon)}(t)| \leq \frac{\kappa}{\epsilon^2}$ and $|D^2\psi_{(\kappa,\epsilon)}(t_1) - D^2\psi_{(\kappa,\epsilon)}(t_2)| \leq 2\kappa\frac{|t_1-t_2|}{\epsilon^3}$, $\forall t, t_1, t_2 \in \mathbb{R}$. Further we have $\psi_{(\kappa,\epsilon)}(0) = -\kappa \ln \epsilon < \beta\sqrt{\kappa} \to 0$ and $D^2\psi_{(\kappa,\epsilon)}(0) = \frac{\kappa}{\epsilon^2} \geq \frac{\alpha}{\sqrt{\epsilon}} \to \infty$ for $(\kappa, \epsilon) \to (0, 0)$.

We must also choose the penalty functions in relation to some feasible point z_f. Hence we assume:

(PQ1) For each $i \in \{1, \ldots, m\}$ we either have

$$g_i(z_f) \leq \varphi_i - \delta \text{ a.e. in } \Omega_i \tag{2.3}$$

for some positive real $\delta > 0$ or the conditions

$$0 \in \operatorname{dom} \psi_{i,q}, \forall q \in \dot{Q}, \quad \lim_{q \to \bar{q}} \psi_{i,q}(0) = 0 \tag{2.4}$$

are fulfilled.

We notice that (PQ1) is fulfilled if (2.4) holds for all $i \in \{1, \ldots, m\}$, i.e. $\psi_{i,q}$ converges pointwise to $\iota_{(-\infty,0]}$ on the whole real axis for every constraint.

In what follows we will show that by the assumptions (Q1),(Q2) and (PQ1) all the assumptions of Theorem 2.1 are fulfilled. We start our analysis with the following lemma:

LEMMA 2.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let $\psi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c. convex function. Then the mapping $\rho : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$,*

$$\rho(u) = \int_\Omega \psi(u(x))\, d\mu(x)$$

is proper, convex and l.s.c.

Proof. Since ψ is l.s.c, for every real c the set $\{t : \psi(t) \leq c\}$ is closed and therefore measurable. Further, by convexity, the set $\psi^{-1}(\{\infty\})$ can be written as the union of two intervals and is therefore also measurable. This shows measurability of ψ and hence for every $u \in L^1(\Omega)$, the function $x \to \psi(u(x))$ is measurable as composition of two measurable functions (see, e.g.,[5]).

Since ψ is a proper l.s.c. convex function on $\mathbb{R}$, it can be expressed as the pointwise supremum of a sequence of affine functions (α_n). Then for every n the function $\beta_n = \max_{1\leq i\leq n} \alpha_i$ is convex and Lipschitz on $\mathbb{R}$ with some modulus L_n. Hence we obtain the bound $|\beta_n(u(x))| \leq |\beta_n(0)| + L_n|u(x)|$ showing integrability of $x \to \beta_n(u(x))$ for every $u \in L^1(\Omega)$. Further, (β_n) is an increasing sequence of functions converging pointwise to ψ. Using Beppo Levi's Theorem we conclude

$$\lim_{n\to\infty} \int_\Omega (\beta_n(u(x)) - \beta_1(u(x)))\, d\mu(x) = \int_\Omega (\psi(u(x)) - \beta_1(u(x)))\, d\mu(x)$$

showing that $\int_\Omega \psi(u(x))\, d\mu(x)$ is well defined. Convexity of ρ is obvious and it remains to show that ρ is l.s.c. The mapping $\rho_n : L^1(\Omega) \to \mathbb{R}$, $\rho_n(u) = \int_\Omega \beta_n(u(x))\, d\mu(x)$ is continuous since

$$|\rho_n(u_1) - \rho_n(u_2)| \leq \int_\Omega L_n|u_1(x) - u_2(x)|\, d\mu(x) = L_n\|u_1 - u_2\|_{L^1(\Omega)}.$$

Using the relation $\rho(u) = \sup_n \rho_n(u)$, we obtain that for any real c the set

$$\{u \in L^1(\Omega) \,:\, \rho(u) \leq c\} = \bigcap_n \{u \in L^1(\Omega) \,:\, \rho_n(u) \leq c\}$$

is the intersection of closed sets and therefore closed. Thus lower semi-continuity of ρ follows. ☐

LEMMA 2.3. *Let $i \in \{i, \ldots, m\}$ and let $\rho : L^{r_i}(\Omega_i) \to \mathbb{R} \cup \{\infty\}$ be a convex l.s.c function. Then the mapping $\zeta : Z \to \mathbb{R} \cup \{\infty\}$, $\zeta(z) := \rho(g_i(z) - \varphi_i)$ is l.s.c. on Z_E. If ζ is convex on Z_E, then it is also weakly l.s.c. on Z_E.*

Proof. Let $\gamma \in \mathbb{R}$ be fixed and consider the set $N_\gamma = \{z \in Z_E \,:\, \zeta(z) \leq \gamma\}$. Since ρ is convex l.s.c. the set $U_\gamma = \{u \in L^{r_i}(\Omega_i) \,:\, \rho(u) \leq \gamma\}$ is convex and closed and hence by (P4), the set $N_\gamma = \{z \in Z_E \,:\, g_i(z) \in \varphi_i + U_\gamma\}$ is closed. Hence ζ is l.s.c. Using Corollary 2.2 in [3, Chapter 1] we obtain that ζ is weakly l.s.c. on Z_E when ζ is convex on Z_E. ☐

COROLLARY 2.4. *For each $q \in \dot{Q}$ and each $i \in \{1,\dots,m\}$ the function $\Psi_{i,q} : Z \to \mathbb{R} \cup \{\infty\}$, $\Psi_{i,q}(z) := \int_{\Omega_i} \psi_{i,q}(g_i(z)(x) - \varphi_i(x))\, d\mu_i(x)$ is convex and weakly l.s.c. on Z_E.*

Proof. For every $z_1, z_2 \in Z_E$, $t \in [0,1]$ we obtain from (P4) and (Q1)

$$\begin{aligned}
&\psi_{i,q}(g_i(tz_1 + (1-t)z_2)(x) - \varphi_i(x)) \\
&\le \psi_{i,q}(t(g_i(z_1)(x) - \varphi_i(x)) + (1-t)(g_i(z_1)(x) - \varphi_i(x))) \\
&\le t\psi_{i,q}(g_i(z_1)(x) - \varphi_i(x)) + (1-t)\psi_{i,q}(g_i(z_1)(x) - \varphi_i(x))
\end{aligned}$$

for almost all $x \in \Omega_i$. Convexity of $\Psi_{i,q}$ on Z_E follows and the assertion follows from Lemma 2.2, Lemma 2.3 and the fact that $L^{r_i}(\Omega_i)$ is continuously embedded in $L^1(\Omega_i)$. □

In what follows we set $\tilde{a}_{i,q} = \psi_{i,q}(-1) + \psi'_{i,q}(-1,1)$, $b_{i,q} = \psi'_{i,q}(-1,1)$ for each i and each $q \in \dot{Q}$. By the monotonicity of $\psi_{i,q}$ we have $b_{i,q} \ge 0$ and from (Q2) and the convexity of $\psi_{i,q}$ we obtain $0 \le \lim_{q\to\bar{q}} \tilde{a}_{i,q} = \lim_{q\to\bar{q}} b_{i,q} \le \lim_{q\to\bar{q}} 2(\psi_{i,q}(-\frac{1}{2}) - \psi_{i,q}(-1)) = 0$. Further, $\alpha_{i,q}(t) = \tilde{a}_{i,q} + b_{i,q}t = \psi_{i,q}(-1) + \psi'_{i,q}(-1,1)(t+1)$ is an affine minorant of $\psi_{i,q}$.

By Lemma 2.3 the function $z \to \int_{\Omega_i} (g_i(z)(x) - \varphi_i(x))\, d\mu_i(x)$ is convex and l.s.c. on Z_E and hence we can find some affine minorant $\gamma_i + \langle \eta_i^*, z\rangle$, $\gamma_i \in \mathbb{R}$, $\eta_i^* \in Z_E^*$. Then we obtain for every $z \in Z_E$

$$\begin{aligned}
\Psi_{i,q}(z) &\ge \int_{\Omega_i} (\tilde{a}_{i,q} + b_{i,q}(g_i(z)(x) - \varphi_i(x)))\, d\mu_i(x) \\
&\ge (\tilde{a}_{i,q}\mu_i(\Omega_i) + b_{i,q}\gamma_i) + b_{i,q}\langle \eta_i^*, z\rangle \\
&\ge a_{i,q} + b_{i,q}\langle \eta_i^*, z\rangle, \qquad (2.5)
\end{aligned}$$

where $a_{i,q} := -|\tilde{a}_{i,q}|\mu_i(\Omega_i)) - |b_{i,q}\gamma_i|$. Note that we have $\lim_{q\to\bar{q}} a_{i,q} = \lim_{q\to\bar{q}} b_{i,q} = 0$.

LEMMA 2.5. *Assume that the assumptions (P1)-(P4), (Q1),(Q2) and (PQ1) are fulfilled. Then for each $i \in \{1,\dots,m\}$ we have $z_f \in \operatorname{dom} \Psi_{i,q_i}$. Further, for each $\hat{z} \in Z_E$ feasible for the problem (P) there is some family $(\hat{z}_q)_{q\in\dot{Q}}$ such that $\lim_{q\to\bar{q}} \hat{z}_q = \hat{z}$ and $\limsup_{q\to\bar{q}} \Psi_{i,q}(\hat{z}_q) \le 0$, $\forall i$.*

Proof. In order to show $z_f \in \operatorname{dom} \Psi_{i,q_i}$ note that by the monotonicity of $\psi_{i,q}$ in case of condition (2.3) we have $\Psi_{i,q}(z_f) \le \int_{\Omega_i} \psi_{i,q}(-\delta)\, d\mu_i(x) \le \infty$ and in case of condition (2.4) we have $\Psi_{i,q}(z_f) \le \int_{\Omega_i} \psi_{i,q}(0)\, d\mu_i(x) \le \infty$.

Now let the set $\{1,\dots,m\}$ be partitioned into a set I_1 containing indices such that (2.3) holds and a set I_2 of indices fulfilling (2.4). By (Q2) we can find for every $n \in N$ some neighborhood U_n of $\bar{q}$ such that for every $q \in U_n$ we have $\psi_{i,q}(-\frac{\delta}{n}) \le \frac{1}{n}$, $i \in I_1$ and $\psi_{i,q}(0) \le \frac{1}{n}$, $i \in I_2$. For each $q \in \dot{Q}$ we define

$$\tau_q = \begin{cases} \inf\{\frac{1}{n} : q \in U_n\} & \text{if } q \in \bigcup_n U_n \\ 1 & \text{otherwise.} \end{cases}$$

Now let $\hat{z} \in Z_E$ be feasible for the problem (P) and set $\hat{z}_q = \tau_q z_f + (1-\tau_q)\hat{z}$. Since $\tau_q \le \frac{1}{n}$ for $q \in U_n$ we have $\lim_{q\to\bar{q}} \hat{z}_q = \hat{z}$. Let $\epsilon > 0$ be arbitrarily fixed, chose $n_\epsilon > \frac{1}{\epsilon}$ and fix some $q \in U_{n_\epsilon}$. We consider first the case $i \in I_1$: If $\tau_q = 0$ then we have $\psi_{i,q}(-\frac{\delta}{n}) \le \frac{1}{n}$, for infinitely many n and from the monotonicity and lower semi-continuity of $\psi_{i,q}$ we can deduce $\psi_{i,q}(0) = \psi_{i,q}(-\tau_q\delta) \le 0$. On the other hand, if $\tau_q > 0$ then $\tau_q = \frac{1}{\bar{n}}$ holds for some $\bar{n} \ge n_\epsilon$ and therefore $\psi_{i,q}(-\tau_q\delta) \le \frac{1}{\bar{n}} \le \frac{1}{n_\epsilon} < \epsilon$ follows. We obtain from (P4) that $g_i(\hat{z}_q) - \varphi_i \le \tau_q(g_i(z_f) - \varphi_i) + (1-\tau_q)(g_i(\hat{z}) - \varphi_i) \le -\tau_q\delta$ almost everywhere in Ω_i and by the monotonicity of $\psi_{i,q}$ the estimate $\Psi_{i,q}(\hat{z}_q) \le \int_{\Omega_i} \psi_{i,q}(-\tau_q\delta)\, d\mu_i(x) \le \epsilon\mu_i(\Omega_i)$ follows.

For $i \in I_2$ we have $\psi_{i,q}(0) \le \tau_q \le \frac{1}{n_\epsilon} < \epsilon$ and since $g_i(\hat{z}_q) - \varphi_i \le 0$, we obtain again the bound $\Psi_{i,q}(\hat{z}_q) \le \int_{\Omega_i} \psi_{i,q}(0)\, d\mu_i(x) \le \epsilon\mu_i(\Omega_i)$. Hence we have shown that for every $\epsilon > 0$ we can find some neighborhood U_ϵ of $\bar{q}$ such that for every $q \in U_\epsilon$ we have $\Psi_{i,q}(\hat{z}_q) \le \epsilon\mu_i(\Omega_i)$, $\forall i$ and $\limsup_{q\to\bar{q}} \Psi_{i,q}(\hat{z}_q) \le 0$ follows. □

LEMMA 2.6. *Assume that the assumptions (P1)-(P4), (Q1),(Q2) are fulfilled. For each $\hat{z} \in Z_E$ not feasible for the problem (P) and for each real R there are some neighborhoods $U_{\hat{z}} \in \mathcal{Z}(\hat{z})$*

and $U_q \in \mathcal{Q}$ such that

$$\inf_{q\in U_q} \inf_{z'\in U_{\hat z}} \sum_{i=1}^{m} \Psi_{i,q}(z') \geq R,$$

Proof. Since $\hat z$ is assumed to be non-feasible, there is some index $\hat\imath$ such that the set $\{x \in \Omega_{\hat\imath} : g_{\hat\imath}(\hat z)(x) - \varphi_{\hat\imath} > 0\}$ has positive measure and consequently there is also some $\epsilon > 0$ such that the set $A_\epsilon := \{x \in \Omega_{\hat\imath} : g_{\hat\imath}(\hat z)(x) - \varphi_{\hat\imath} > \epsilon\}$ has measure $\mu > 0$. Using (P4) we see that $\zeta_\epsilon : Z \to \mathbb{R}$, $\zeta_\epsilon(z) = \int_{A_\epsilon} (g_{\hat\imath}(z)(x) - \varphi_{\hat\imath}(x))\, d\mu_{\hat\imath}(x)$ is convex on Z_E and by Lemma 2.3 we conclude that ζ_ϵ is weakly l.s.c. on Z_E. Since $\zeta_\epsilon(\hat z) \geq \epsilon\mu$, the set $U_{\hat z} := \{z' \in Z_E : \zeta_\epsilon(z') > \frac{\mu\epsilon}{2}, \langle \eta_i^*, z'\rangle > \langle \eta_i^*, \hat z\rangle - 1, \forall i\}$ is a weak neighborhood of $\hat z$ in Z_E. It follows that for every $z' \in U_{\hat z}$ we have $\int_{A_{\epsilon,z'}} (g_{\hat\imath}(z')(x) - \varphi_{\hat\imath}(x))\, d\mu_{\hat\imath}(x) \geq \frac{\mu\epsilon}{4}$ where $A_{\epsilon,z'} := \{x \in A_\epsilon : g_{\hat\imath}(z')(x) - \varphi_{\hat\imath}(x) \geq \frac{\epsilon}{4}\}$. By (Q2) we can find for each $N > 1$ some neighborhood $U_N \in \mathcal{Q}$ such that $\psi_{\hat\imath,q}(\frac{\epsilon}{4}) > N$ and $\psi_{\hat\imath,q}(-1) < 1$ and by convexity of $\psi_{\hat\imath,q}$ we have $\psi_{\hat\imath,q}(t) \geq N + \gamma_N(t - \frac{\epsilon}{4}) \geq \gamma_N t$, $\forall t \geq \frac{\epsilon}{4}$, where $\gamma_N := \frac{N-1}{1+\epsilon/4}$. For arbitrary large R we can choose N such that $(\gamma_N - 1)\frac{\mu\epsilon}{4} > R+1$ and we can choose a neighborhood $U_q \subset U_N$ such that $b_{\hat\imath,q} < 1$ and $\sum_{i=1}^m a_{i,q} + b_{i,q}(\langle \eta_i^*, \hat z\rangle - 1) > -1$, $\forall q \in U_q$.

From (2.5) we obtain for each $q \in U_q$ and each $z' \in U_{\hat z}$

$$\begin{aligned}
\Psi_{\hat\imath,q}(z') &= \int_{A_{\epsilon,z'}} \psi_{\hat\imath,q}(g_{\hat\imath}(z')(x) - \varphi_{\hat\imath}(x))\, d\mu_{\hat\imath}(x) \\
&\qquad + \int_{\Omega_i \setminus A_{\epsilon,z'}} \psi_{\hat\imath,q}(g_{\hat\imath}(z')(x) - \varphi_{\hat\imath}(x))\, d\mu_{\hat\imath}(x) \\
&\geq \int_{A_{\epsilon,z'}} \gamma_N(g_{\hat\imath}(z')(x) - \varphi_{\hat\imath}(x))\, d\mu_{\hat\imath}(x) \\
&\qquad + \int_{\Omega_i \setminus A_{\epsilon,z'}} (\tilde a_{\hat\imath,q} + b_{\hat\imath,q}(g_{\hat\imath}(z')(x) - \varphi_{\hat\imath}(x)))\, d\mu_{\hat\imath}(x) \\
&= \int_{A_{\epsilon,z'}} (\gamma_N - b_{\hat\imath,q})(g_{\hat\imath}(z')(x) - \varphi_{\hat\imath}(x))\, d\mu_{\hat\imath}(x) + \tilde a_{\hat\imath,q}\mu_{\hat\imath}(\Omega_i \setminus A_{\epsilon,z'}) \\
&\qquad + \int_{\Omega_i} b_{\hat\imath,q}(g_{\hat\imath}(z')(x) - \varphi_{\hat\imath}(x))\, d\mu_{\hat\imath}(x) \\
&\geq (\gamma_N - b_{\hat\imath,q})\frac{\mu\epsilon}{4} + a_{\hat\imath,q} + b_{\hat\imath,q}(\langle \eta_{\hat\imath}^*, \hat z\rangle - 1) > R + 1 + a_{\hat\imath,q} + b_{\hat\imath,q}(\langle \eta_{\hat\imath}^*, \hat z\rangle - 1)
\end{aligned}$$

and $\Psi_{i,q}(z') \geq a_{i,q} + b_{i,q}(\langle \eta_i^*, \hat z\rangle - 1)$, $\forall i \neq \hat\imath$. Summing up we obtain $\sum_{i=1}^m \Psi_{i,q}(z') \geq R$ and this completes the proof. □

COROLLARY 2.7. *Assume that the assumptions (P1)-(P4), (Q1),(Q2) and (PQ1) are fulfilled. Then all the assumptions of Theorem 2.1 are fulfilled. In particular, for each $q \in \dot Q$ the problem (P_q) has a unique solution $\bar z_q$ and $\lim_{q\to\bar q} \bar z_q = \bar z$.*

3. Error estimates. Let $\hat z, z \in Z_E$ and $i \in \{1,\dots,m\}$. Then, by (P4) there is some measurable function $h : \Omega_i \to \mathbb{R}$ such that for every sequence $(t_n) \downarrow 0$ of positive numbers converging to 0 we have

$$h(x) = \lim_n t_n^{-1}(g_i(\hat z + t_n(z - \hat z))(x) - g_i(\hat z))\ \mu_i\text{-a.e. in } \Omega_i.$$

We denote such a function h by $g_i'(\hat z, z - \hat z)$. Note that $g_i'(\hat z, z - \hat z)$ is uniquely defined only up to a set of measure 0 and that $g_i(z)(x) \geq g_i(\hat z)(x) + g_i'(\hat z, z - \hat z)(x)$ for almost all $x \in \Omega_i$.

LEMMA 3.1. *Let $i \in \{1,\dots,m\}$, $q \in Q$ be fixed and let $\hat z \in \operatorname{dom}\Psi_{i,q} \cap Z_E$, $z \in Z_E$ be such that $\hat z + t_0(z - \hat z) \in \operatorname{dom}\Psi_{i,q}$ for some positive real $t_0 > 0$. Then*

$$\lim_{t\downarrow 0} \frac{\Psi_{i,q}(\hat z + t(z - \hat z)) - \Psi_{i,q}(\hat z)}{t} \tag{3.1}$$

$$= \int_{\Omega_i} \psi^{\#}_{i,q}(g_i(\hat z)(x) - \varphi_i(x), g_i'(\hat z, z-\hat z)(x))\; d\mu_i(x) \in \mathbb{R}\cup\{-\infty\}$$

holds, where

$$\psi^{\#}_{i,q}(u,h) = \begin{cases} \psi'_{i,q}(u,h) & \text{if } u\in \operatorname{int}\operatorname{dom}\psi_{i,q}\\ \lim_{t\downarrow 0}\frac{\psi_{i,q}(u+th)-\psi_{i,q}(u)}{t} \in \mathbb{R}\cup\{-\infty\} & \text{if } u\in \operatorname{bd}\operatorname{dom}\psi_{i,q},\ h<0\\ 0 & \text{otherwise}\end{cases}$$

Proof. Let $(t_n)_{n\in\mathbb{N}_0}\downarrow 0$ and

$$s_n(x) := \frac{\psi_{i,q}(g_i(\hat z + t_n(z-\hat z))(x) - \varphi_i(x)) - \psi_{i,q}(g_i(\hat z)(x)-\varphi_i(x))}{t_n}.$$

In view of (P4) and (Q1) the sequence $(s_n(x))$ is monotonically decreasing for almost all $x\in\Omega_i$. Further we have $\lim_n s_n(x) = \psi'_{i,q}(g_i(\hat z)(x)-\varphi_i(x), g_i'(\hat z, z-\hat z)(x))$ for almost all $x\in\Omega_i$ fulfilling $g_i(\hat z)(x)-\varphi_i(x)\in \operatorname{int}\operatorname{dom}\psi_{i,q}$. Since $\hat z\in\operatorname{dom}\Psi_{i,q}$ we have $g_i(\hat z)(x)-\varphi_i(x)\in\operatorname{bd}\operatorname{dom}\psi_{i,q}$ for almost all $x\in M := \{x\in\Omega_i : g_i(\hat z)(x)-\varphi_i(x)\notin \operatorname{int}\operatorname{dom}\psi_{i,q}$. By virtue of $\hat z + t_0(z-\hat z)\in\operatorname{dom}\Psi_{i,q}$ it follows that $\hat z + t_n(z-\hat z)\in\operatorname{dom}\Psi_{i,q}$, $\forall n$ and consequently $g_i(\hat z + t_n(z-\hat z))(x)-\varphi_i(x)\in \operatorname{dom}\psi_{i,q}$, $\forall n$ for almost all $x\in\Omega_i$. This implies $g_i'(\hat z, z-\hat z)(x)\le 0$ for almost all $x\in M$ and $g_i(\hat z+t_n(z-\hat z))(x) = g_i(\hat z)(x)$, $\forall n$ for almost all $x\in M$ with $g_i'(\hat z, z-\hat z)(x)=0$. Therefore we also have $\lim_n s_n(x) = \psi^{\#}_{i,q}(g_i(\hat z)(x)-\varphi_i(x), g_i'(\hat z, z-\hat z)(x))$ for almost all $x\in M$. Since s_0 is integrable we can apply Beppo-Levi's Theorem to obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{\Psi_{i,q}(\hat z + t_n(z-\hat z)) - \Psi_{i,q}(\hat z)}{t_n} &= \lim_{n\to\infty}\int_{\Omega_i} s_n(x)\; d\mu_i(x)\\ &= \int_{\Omega_i}\psi^{\#}_{i,q}(g_i(\hat z)(x)-\varphi_i(x), g_i'(\hat z, z-\hat z)(x))\; d\mu_i(x)\end{aligned}$$

and the assertion follows. ◻

From the definition and (Q1) we obtain readily that for each fixed u the mapping $\psi^{\#}_{i,q}(u,\cdot)$ is positively homogenous, i.e. $\psi^{\#}_{i,q}(u,\lambda h) = \lambda\psi^{\#}_{i,q}(u,h), \forall\lambda>0, h\in\mathbb{R}$. Further we have $\psi^{\#}_{i,q}(u,h)\ge 0$, $\forall h\ge 0$ and $\psi^{\#}_{i,q}(u,h)\le 0$, $\forall h<0$. From this we also obtain the monotonicity property

$$\psi^{\#}_{i,q}(u,h_1)\le \psi^{\#}_{i,q}(u,h_2),\ \forall h_1\le h_2. \tag{3.2}$$

Note that for $u\in\operatorname{bd}\operatorname{dom}\psi_{i,q}$ we can have $\psi^{\#}_{i,q}(u,h) = -\infty$ and also the limit in (3.1) can be $-\infty$. However, the latter is not possible for $\hat z = \bar z_q$ since $\bar z_q$ minimizes J_q. The following assumption guarantees the applicability of Lemma 3.1 in case of a feasible point z

(Q3') For each $i\in\{1,\dots,m\}$ and each $q\in\dot Q$ there are nonnegative constants $c_{i,q}, d_{i,q}$ such that

$$\psi_{i,q}(\frac{t}{2}) \le c_{i,q}\psi_{i,q}(t) + d_{i,q},\ \forall t\in[-1,0)$$

For instance, Assumption (Q3') is fulfilled if $0\in\operatorname{dom}\psi_{i,q}$ or $\psi_{i,q}$ grows like $-\ln(-t)$ or $(-t)^{-p}$ for $t\to 0_-$ as in the case of the usual barrier functions.

COROLLARY 3.2. *Assume that the assumptions (P1)-(P4), (Q1)-(Q3') and (PQ1) are fulfilled. Let $q\in\dot Q$ be fixed and let $z\in Z_E$ be feasible for the problem (P). Then J_q is directionally differentiable at $\bar z_q$ in direction $z-\bar z_q$ with directional derivative*

$$J_q'(\bar z_q, z-\bar z_q) = J'(\bar z_q, z-\bar z_q) + \sum_{i=1}^m\int_{\Omega_i}\psi^{\#}_{i,q}(g_i(\bar z_q)(x)-\varphi_i(x), g_i'(\bar z_q, z-\bar z_q)(x))\; d\mu_i(x)$$

Proof. The assertion follows from Lemma 3.1 if we can show that $z' = \bar z_q + \frac{1}{2}(z-\bar z_q)\in\operatorname{dom}J_q$. For each i we obtain for almost all $x\in\Omega_i$ by virtue of (P4),(Q1), (Q3') and the feasibility of z

$$\psi_{i,q}(g_i(z')(x)-\varphi_i(x)) \le \psi_{i,q}(\frac{1}{2}(g_i(\bar z_q)(x)-\varphi_i(x))+\frac{1}{2}(g_i(z)(x)-\varphi_i(x)))$$
$$\le \begin{cases} \psi_{i,q}(g_i(\bar z_q)(x)-\varphi_i(x)) & \text{if } \psi_{i,q}(g_i(\bar z_q)(x)-\varphi_i(x)) \ge 0 \\ c_{i,q}\psi_{i,q}(g_i(\bar z_q)(x)-\varphi_i(x))+d_{i,q} & \text{if } -1 \le \psi_{i,q}(g_i(\bar z_q)(x)-\varphi_i(x)) \le 0 \\ \psi_{i,q}(-\frac{1}{2}) & \text{otherwise} \end{cases}$$

Hence the integrand in $\int_{\Omega_i} \psi_{i,q}(g_i(z')(x)-\varphi_i(x))\, d\mu_i(x)$ is bounded above by an integrable function. We can conclude that $\Psi_{i,q}(z')$ is finite and therefore $z' \in \operatorname{dom} J_q$ holds. □

THEOREM 3.3. *Suppose that (P1)-(P4), (Q1)-(Q3') and (PQ1) hold. Then there is a neighborhood $N \in \mathcal{Q}$ and a constant L such that for each $q \in N$ we have*

$$\frac{\alpha}{2}\|\bar z_q - \bar z\|_Z^2 \le L\text{dist}\,(\bar z_q, \mathcal{F}) + \sum_{i=1}^m \int_{\Omega_i} \psi^{\#}_{i,q}(g_i(\bar z_q)(x)-\varphi_i(x), -(g_i(\bar z_q)(x)-\varphi_i(x)))\, d\mu_i(x),$$

where $\mathcal{F}$ denotes the feasible region of the problem (P).

Proof. We always have $J(\bar z) \ge J(\bar z_q) + J'(\bar z_q, \bar z - \bar z_q) + \frac{\alpha}{2}\|\bar z - \bar z_q\|_Z^2$ by (P3). Since $\bar z_q$ minimizes J_q we obtain by Corollary 3.2 $J_q'(\bar z_q, \bar z - \bar z_q) \ge 0$ and consequently

$$J(\bar z) \ge J(\bar z_q) + \frac{\alpha}{2}\|\bar z - \bar z_q\|_Z^2 - \sum_{i=1}^m \int_{\Omega_i} \psi^{\#}_{i,q}(g_i(\bar z_q)(x)-\varphi_i(x), g_i'(\bar z_q, \bar z - \bar z_q)(x))\, d\mu_i(x).$$

Because $\bar z$ is feasible we have $g_i'(\bar z_q, \bar z - \bar z_q)(x) \le g_i(\bar z)(x) - \varphi_i(x) - (g_i(\bar z_q) - \varphi_i(x)) \le -(g_i(\bar z_q) - \varphi_i(x))$ for almost all $x \in \Omega_i$ and from (3.2) we obtain

$$J(\bar z) \ge J(\bar z_q) + \frac{\alpha}{2}\|\bar z - \bar z_q\|_Z^2 - \sum_{i=1}^m \int_{\Omega_i} \psi^{\#}_{i,q}(g_i(\bar z_q)(x)-\varphi_i(x), -(g_i(\bar z_q)(x)-\varphi_i(x)))\, d\mu_i(x).$$

Now let z_q^f denote the projection of $\bar z_q$ onto the closed convex set $\mathcal{F}$. Since $\|\bar z_q - z_q^f\|_Z \le \|\bar z_q - \bar z\|_Z$ we have $\|z_q^f - \bar z\|_Z \le 2\|\bar z_q - \bar z\|_Z$ and from Corollary 2.7 together with the Lipschitz continuity of J near $\bar z$ it follows that there is some neighborhood $N \in \mathcal{Q}$ such that for all $q \in N$ both $\bar z_q$ and z_q^f belong to a neighborhood of $\bar z$ where J is Lipschitz with modulus L. Hence we have $J(\bar z) \le J(z_q^f) \le J(\bar z_q) + L\|\bar z_q - z_q^f\|_Z = J(\bar z_q) + L\text{dist}\,(\bar z_q, \mathcal{F})$ and together with the inequality above the assertion follows. □

If we can estimate the distance of $\bar z_q$ to the feasible set, Theorem 3.3 provides us with a effective tool for estimating the quality of our approximate solution $\bar z_q$

Note that in case of the combined logarithmic-quadratic penalty function given by (2.2) we have

$$\psi^{\#}_{i,(\kappa,\epsilon)}(t,-t) = \begin{cases} \kappa & \text{if } t \le -\epsilon \\ -\kappa(\frac{2t}{\epsilon} + \frac{t^2}{\epsilon^2}) \le \kappa & \text{if } t > -\epsilon \end{cases}.$$

and therefore the a-priori bound

$$\int_{\Omega_i} \psi^{\#}_{i,(\kappa,\epsilon)}(g_i(\bar z_{(\kappa,\epsilon)})(x)-\varphi_i(x), -(g_i(\bar z_{(\kappa,\epsilon)})(x)-\varphi_i(x)))\, d\mu_i(x) \le \kappa\mu_i(\Omega_i)$$

is available. If we want to approximate the exact solution $\bar z$ within some given precision this formula allows us to choose κ in advance and then to adjust ϵ such that dist $(\bar z_{(\kappa,\epsilon)}, \mathcal{F})$ is sufficiently small.

4. Duality. In this section we demonstrate that assuming a constraint qualification condition ensuring the existence of multipliers we can also approximate these multipliers by our generalized penalty method. To do this we need the following additional assumption:

(Q3) For each $i \in \{1,\dots,m\}$ and each $q \in \dot Q$ we assume that $\psi_{i,q}$ is differentiable on $\mathbb{R}$ and $\limsup_{t\to\infty} t^{-r_i}\psi_{i,q}(t) < \infty$. Further we assume $\lim_{q\to\bar q} \psi_{i,q}(0) = 0$.

Note that (Q3) implies the assumptions (PQ1) and (Q3'). Further, (Q3) together with (Q1) implies the bound $|\psi_{i,q}(t)| \le C_{i,q}(1+|t|^{r_i})$, $\forall t$. Consequently, for each $u \in L^{r_i}(\Omega_i)$ the function $x \to \psi_{i,q}(u(x))$ is μ_i-integrable. Since $L^{r_i}(\Omega_i)$ is continuously embedded in $L^1(\Omega_i)$ we obtain from Lemma 2.2 that the mapping $\rho_{i,q}(u) := \int_{\Omega_i} \psi_{i,q}(u(x))\, d\mu_i(x)$ is l.s.c. convex and real-valued on $L^{r_i}(\Omega_i)$. By applying Corollaries 2.4, 2.5 in [3, Chapter 1] we can deduce that $\rho_{i,q}$ is even continuous and locally Lipschitz and hence also everywhere directionally differentiable. Similar arguments as in the proof of Lemma 3.1 show, that

$$\rho'_{i,q}(u,h) = \int_{\Omega_i} \psi'_{i,q}(u(x), h(x))\, d\mu_i(x).$$

Lipschitz continuity of $\rho_{i,q}$ near u with some modulus L implies the bound $\rho'_{i,q}(u,h) \le L\|h\|_{L^{r_i}(\Omega_i)}$ and together with the differentiability of $\psi_{i,q}$ we deduce that $h \to \rho'_{i,q}(u,h)$ is a bounded linear functional on $L^{r_i}(\Omega_i)$. We define now for each i and each $q \in \dot{Q}$ the linear functional $l^*_{i,q} \in L^{r_i}(\Omega_i)^*$ by

$$\langle l^*_{i,q}, h\rangle := \rho'_{i,q}(g_i(\bar{z}_q) - \varphi_i, h) = \int_{\Omega_i} \psi'_{i,q}(g_i(\bar{z}_q)(x) - \varphi_i(x), h(x))\, d\mu_i(x).$$

Note that due to the monotonicity of $\psi_{i,q}$ we have $\langle l^*_{i,q}, h\rangle \le 0$ for every $h \in L^{r_i}(\Omega_i)$ satisfying $h \le 0$ μ_i-a.e. in Ω_i and therefore by virtue of (P4) the mapping $z \to \langle l^*_{i,q}, g_i(z)\rangle$ is convex on Z_E.

LEMMA 4.1. *For each $q \in \dot{Q}$ the point $\bar{z}_q$ is the unique solution of the problem*

$$\min_{z\in Z_E} \tilde{J}_q(z) := J(z) + \sum_{i=1}^m \langle l^*_{i,q}, g_i(z)\rangle.$$

Moreover we have

$$\lim_{q\to\bar{q}} \sum_{i=1}^m \langle l^*_{i,q}, g_i(\bar{z}) - g_i(\bar{z}_q)\rangle = \lim_{q\to\bar{q}} \sum_{i=1}^m \langle l^*_{i,q}, \varphi_i - g_i(\bar{z}_q)\rangle = 0 \tag{4.1}$$

and

$$\limsup_{q\to\bar{q}} \sum_{i=1}^m \langle l^*_{i,q}, u_i - g_i(\bar{z})\rangle \le 0 \tag{4.2}$$

for each $(u_1,\dots,u_m) \in \prod_{i=1}^m L^{r_i}(\Omega_i)$ such that $u_i \le \varphi_i$ μ_i-a.e. in Ω_i.

Proof. The arguments above show that for each q the function J_q is continuous on Z_E and hence also directionally differentiable. Since $\bar{z}_q$ solves the problem (P_q) we obtain from (P2), (P4), (3.2) and Lemma 3.1 that

$$\begin{aligned}
&0 \le J'_q(\bar{z}_q, z - \bar{z}_q)\\
&\quad = J'(\bar{z}_q, z-\bar{z}_q) + \sum_{i=1}^m \int_{\Omega_i} \psi'_{i,q}(g_i(\bar{z}_q)(x) - \varphi_i(x), g'_i(z_q, z - \bar{z}_q)(x))\, d\mu_i(x)\\
&\quad < J(z) - J(z_q) + \sum_{i=1}^m \int_{\Omega_i} \psi'_{i,q}(g_i(\bar{z}_q)(x) - \varphi_i(x), g_i(z)(x) - g_i(\bar{z}_q)(x))\, d\mu_i(x) \qquad (4.3)\\
&\quad = \tilde{J}_q(z) - \tilde{J}_q(\bar{z}_q)
\end{aligned}$$

holds for each $z \in Z_E$, $z \ne z_q$. Hence z_q is the unique minimizer of $\tilde{J}_q$ on Z_E. Taking into account relation (4.3) with $z = \bar{z}$ together with the feasibility of $\bar{z}$ and (Q1) we obtain

$$J(\bar{z}_q) - J(\bar{z}) \le \sum_{i=1}^m \int_{\Omega_i} \psi'_{i,q}(g_i(\bar{z}_q)(x) - \varphi_i(x), g_i(\bar{z})(x) - g_i(\bar{z}_q)(x))\, d\mu_i(x)$$

$$= \sum_{i=1}^{m} \int_{\Omega_i} \psi'_{i,q}(g_i(\bar{z}_q)(x) - \varphi_i(x), g_i(\bar{z})(x) - \varphi_i(x))\, d\mu_i(x)$$

$$+ \sum_{i=1}^{m} \int_{\Omega_i} \psi'_{i,q}(g_i(\bar{z}_q)(x) - \varphi_i(x), \varphi_i(x) - g_i(\bar{z}_q)(x))\, d\mu_i(x)$$

$$\leq \sum_{i=1}^{m} \int_{\Omega_i} \psi'_{i,q}(g_i(\bar{z}_q)(x) - \varphi_i(x), \varphi_i(x) - g_i(\bar{z}_q)(x))\, d\mu_i(x)$$

$$\leq \sum_{i=1}^{m} \int_{\Omega_i} (\psi_{i,q}(0) - \psi_{i,q}(g_i(\bar{z}_q)(x) - \varphi_i(x)))\, d\mu_i(x)$$

$$\leq \sum_{i=1}^{m} (\psi_{i,q}(0)\mu_i(\Omega_i) - (a_{i,q} + b_{i,q}\langle \eta_i^*, \bar{z}_q\rangle))$$

Hence, relation (4.1) follows by taking into account $\lim_{q\to\bar{q}} z_q = \bar{z}$, $\lim_{q\to\bar{q}} |\psi_{i,q}(0)|+|a_{i,q}|+|b_{i,q}| = 0$ and the continuity of J on Z_E . The inequality (4.2) now follows from (4.1) together with the relation

$$\psi'_{i,q}(g_i(\bar{z}_q)(x) - \varphi_i(x), u_i(x) - g_i(\bar{z})(x)) \leq \psi'_{i,q}(g_i(\bar{z}_q)(x) - \varphi_i(x), \varphi_i(x) - g_i(\bar{z})(x)),$$

valid almost everywhere in Ω_i. □

We will show below that every weak-∗ limit point of the linear functionals l_{i,q_i} is a solution of some dual problem. However, such limit points will not exist in $L^{r_i}(\Omega_i)^*$ generally, but eventually in some other dual space. The existence of such limit points is closely related to some constraint qualification. To present a comprehensive result it is necessary to consider bilateral constraints, i.e. we consider the problem

$$(\hat{P}) \qquad \min_{z\in Z} J(z)$$

subject to

$$Ez = 0,$$
$$\underline{\varphi}_j \leq A_j z \leq \overline{\varphi}_j \quad \mu_j\text{-a.e. in } \hat{\Omega}_j,\ j = 1,\dots,m'$$
$$\hat{g}_j(z) \leq \hat{\varphi}_j \quad \mu_j\text{-a.e. in } \hat{\Omega}_j,\ j = m'+1,\dots,m'',$$

where $A_j \in \mathcal{L}(Z, L^{p_j}(\hat{\Omega}_j))$, $j = 1,\dots,m'$ are bounded linear operators, and $\hat{g}_j : Z \to L^{p_j}(\hat{\Omega}_j)$. We assume, that with the setting $m = m' + m''$, $\Omega_{2j-1} = \Omega_{2j} = \hat{\Omega}_j$, $r_{2j-1} = r_{2j} = p_j$, $g_{2j-1} = -g_{2j} = A_j$, $\varphi_{2j-1} = \overline{\varphi}_j$, $\varphi_{2j} = -\underline{\varphi}_j$, $1 \leq j \leq m'$ and $\Omega_{m'+j} = \hat{\Omega}_j$, $r_{m'+j} = p_{m'+j}$, $g_{m'+j} = \hat{g}_j$, $\varphi_{m'+j} = \hat{\varphi}_j$, $m'+1 \leq j \leq m''$ our problem (P) is an equivalent reformulation of the problem $(\hat{P})$ by splitting the bilateral constraints.

Further we assume that there is some Banach space $\hat{Z}$ continuously embedded in Z such that $\bar{z} \in \hat{Z}$ and some Banach space Y continuously embedded in $\prod_{j=1}^{m''} L^{p_j}(\hat{\Omega}_j)$ such that $\hat{G}(\hat{Z}) \subset Y$, where $\hat{G}(z) := (A_1 z, \dots, A_{m'} z, \hat{g}_{m'+1}(z), \dots, g_{m'+m''}(z))$. We also define the closed convex set $C \subset Y$ by

$$C := \left\{ c = (c_1,\dots,c_{m''}) \in Y \;:\; \begin{array}{l} \underline{\varphi}_j \leq c_j \leq \overline{\varphi}_j \quad \mu_j\text{-a.e. in } \hat{\Omega}_j,\ j = 1,\dots,m' \\ c_j \leq \hat{\varphi}_j \quad \mu_j\text{-a.e. in } \hat{\Omega}_j,\ j = m'+1,\dots,m'' \end{array} \right\}.$$

Since E is also continuous as an operator from $\hat{Z}$ into V, the subspace $\hat{Z}_E = Z_E \cap \hat{Z}$ of $\hat{Z}$ induced by the kernel of E is a Banach space. Note that our assumptions imply that $\bar{z}$ is the unique solution for the problem

$$(\hat{P}') \qquad \min_{z\in\hat{Z}} J(z)$$

subject to

$$Ez = 0,$$
$$\hat{G}(z) \in C.$$

For each $q \in Q$ we also define the linear functionals $y_q^* \in (\prod_{j=1}^{m''} L^{p_j}(\hat{\Omega}_j))^* \subset Y^*$ by

$$\langle y_q^*, y\rangle = \sum_{j=1}^{m'} \langle l_{2j-1,q}^* - l_{2j,q}^*, y_j\rangle + \sum_{j=m'+1}^{m''} \langle l_{m'+j,q}^*, y_j\rangle.$$

THEOREM 4.2. *Assume that the problem (P) is an equivalent reformulation of the problem* $(\hat{P})$ *and suppose that assumptions (P1)-(P4), (Q1), (Q2) and (Q3) are fulfilled. Let* $(q^n) \subset \dot{Q}$ *be a sequence converging to* $\bar{q}$ *and assume that* $y^* \in Y^*$ *is a weak-∗ limit point of the sequence* $(y_{q^n}^*)$*. Then* y^* *belongs to the normal cone of* C *at* $\hat{G}(\bar{z})$*, denoted by* $N_C(\hat{G}(\bar{z}))$*, i.e.* $\langle y^*, c - \hat{G}(\bar{z})\rangle \leq 0$*,* $\forall c \in C$*, and* $\bar{z}$ *is the unique solution for the convex problem*

$$\min_{z \in \hat{Z}_E} J(z) + \langle y^*, \hat{G}(z)\rangle \tag{4.4}$$

Proof. For $c \in C$ let $u_{2j-1} = c_j$, $u_{2j} = -c_j$, $j = 1, \dots, m'$ and $u_{m'+j} = c_j$, $j = m' + 1, \dots, m''$.Then u satisfy $u_i \leq \varphi_i$, $1 \leq i \leq m$ and by (4.2) we obtain

$$\begin{aligned}
\langle y^*, c - \hat{G}(\bar{z})\rangle &\leq \limsup_{n\to\infty} \langle y_{q^n}^*, c - \hat{G}(\bar{z})\rangle \\
&= \limsup_{n\to\infty} \Bigg(\sum_{j=1}^{m'} (\langle l_{2j-1,q^n}^*, c_j - A_j\bar{z}\rangle + \langle l_{2j,q^n}^*, A_j\bar{z} - c_j\rangle) \\
&\qquad + \sum_{j=m'+1}^{m''} \langle l_{m'+j,q^n}^*, c_j - \hat{g}_j(\bar{z})\rangle \Bigg) \\
&= \limsup_{n\to\infty} \sum_{i=1}^{m} \langle l_{i,q^n}^*, u_i - g_i(\bar{z})\rangle \leq 0.
\end{aligned}$$

Now let us show that the objective of (4.4) is strictly convex on $\hat{Z}_E$. Let $z_1, z_2 \in \hat{Z}_E$, $\alpha \in [0, 1]$ be fixed. Then there is a subsequence of $(y_{q^n}^*)$, w.l.o.g. the sequence $(y_{q^n}^*)$ itself, such that $\lim_n \langle y_{q^n}^*, \hat{G}(z_i)\rangle = \langle y^*, \hat{G}(z_i)\rangle$, $i = 1, 2$ and $\lim_n \langle y_{q^n}^*, \hat{G}(\alpha z_1 + (1-\alpha)z_2)\rangle = \langle y^*, \hat{G}(\alpha z_1 + (1-\alpha)z_2)\rangle$. Hence we obtain

$$\begin{aligned}
\langle y^*, \hat{G}(\alpha z_1 + (1-\alpha)z_2)\rangle &= \lim_n \sum_{i=1}^{m} \langle l_{i,q^n}^*, g_i(\alpha z_1 + (1-\alpha)z_2)\rangle \\
&\leq \lim_n \sum_{i=1}^{m} (\alpha \langle l_{i,q^n}^*, g_i(z_1)\rangle + (1-\alpha)\langle l_{i,q^n}^*, g_i(z_2)\rangle) \\
&= \alpha\langle y^*, \hat{G}(z_1)\rangle + (1-\alpha)\langle y^*, \hat{G}(z_2)\rangle
\end{aligned}$$

Together with the strict convexity of J on Z_E the strict convexity of the objective function of (4.4) on $\hat{Z}_E \subset Z_E$ follows. Now assume that $\bar{z}$ is not a solution to (4.4). Then there is some point $\hat{z} \in \hat{Z}_E$ and some real $\epsilon > 0$ such that $J(\hat{z}) - J(\bar{z}) + \langle y^*, \hat{G}(\hat{z}) - \hat{G}(\bar{z})\rangle < -3\epsilon$. Further there is a subsequence of $(y_{q^n}^*)$, w.l.o.g. the sequence $(y_{q^n}^*)$ itself, such that for every n we have $\langle y_{q^n}^* - y^*, \hat{G}(\hat{z}) - \hat{G}(\bar{z})\rangle < \epsilon$, $J(\bar{z}) - J(\bar{z}_{q^n}) < \epsilon$ and $\sum_{i=1}^{m} \langle l_{i,q^n}^*, g_i(\bar{z}) - g_i(\bar{z}_{q^n})\rangle < \epsilon$. Consequently we have

$$\begin{aligned}
\tilde{J}_{q^n}(\hat{z}) - \tilde{J}_{q^n}(\bar{z}_{q^n}) &= J(\hat{z}) - J(\bar{z}_{q^n}) + \sum_{i=1}^{m} \langle l_{i,q^n}^*, g_i(\hat{z}) - g_i(\bar{z}_{q^n})\rangle \\
&< J(\hat{z}) - J(\bar{z}) + \langle y^*, \hat{G}(\hat{z}) - \hat{G}(\bar{z})\rangle + 3\epsilon < 0,
\end{aligned}$$

a contradiction to the optimality of $\bar{z}_{q^n}$ for $\tilde{J}_{q^n}$. Hence, $\bar{z}$ solves (4.4) and due to strict convexity of the objective it is the unique solution. □

PROPOSITION 4.3. *Assume that the problem (P) is an equivalent reformulation of the problem $(\hat{P})$ and suppose that assumptions (P1)-(P4), (Q1), (Q2) and (Q3) are fulfilled. Further assume that the mapping $\hat{G}$ is continuous on $\hat{Z}_E$ and*

$$0_Y \in \operatorname{int}(\hat{G}(\hat{Z}_E) - C). \tag{4.5}$$

Then

$$\limsup_{q\to\bar{q}} \|y_q^*\|_{Y^*} < \infty$$

Proof. Due to (P4) and since $\hat{G}$ is continuous on $\hat{Z}_E$ the multifunction $\Theta : \hat{Z}_E \to 2^Y$, $\Theta(z) = \hat{G}(z) - C$ is closed convex. In view of (4.5) we can use the Robinson-Ursescu stability theorem (see, e.g. [2, Theorem 2.83]) to see that Θ is metrical regular near $(\bar{z}, 0)$. Hence there are some radius $r > 0$ and some constant $\gamma > 0$ such that for every $y \in Y$, $\|y\|_Y \le r$ there is some $z_y \in \hat{Z}_E$ such that $\|z_y - \bar{z}\|_{\hat{Z}} \le \gamma \operatorname{dist}(y, \hat{G}(\bar{z}) - C) \le \gamma\|y\|_Y$ and $y \in \hat{G}(z_y) - C$. We may assume that r is so small that J is Lipschitz with some modulus L in the ball $B_{\hat{Z}_E}(\bar{z}, \gamma r)$ around $\bar{z}$ with radius γr. Let $y \in Y$, $\|y\|_Y \le r$ be fixed and let z_y and $c_y \in C$ be such that $y = \hat{G}(z_y) - c_y$. Denote $\epsilon_q = \sum_{i=1}^m \langle l_{i,q}^*, \varphi_i - g_i(\bar{z}_q)\rangle$. Due to (4.3) and the relations between the problem (P) and $(\hat{P})$ we obtain

$$\begin{aligned}
&J(\bar{z}_q) - J(z_y) - \epsilon_q \\
&\le \sum_{i=1}^m \int_{\Omega_i} \psi_{i,q}'(g_i(\bar{z}_q)(x) - \varphi_i(x), g_i(z_y)(x) - g_i(\bar{z}_q)(x))\, d\mu_i(x) - \epsilon_q \\
&= \sum_{i=1}^m \langle l_{i,q}^*, g_i(z_y) - \varphi_i\rangle \\
&= \sum_{j=1}^{m'} (\langle l_{2j-1,q}^*, A_j z_y - \overline{\varphi}_j\rangle + \langle l_{2j,q}^*, \underline{\varphi}_j - A_j z_y\rangle) + \sum_{j=m'+1}^{m''} \langle l_{m'+j,q}^*, \hat{g}_j(z_y) - \hat{\varphi}_j\rangle \\
&= \sum_{j=1}^{m'} (\langle l_{2j-1,q}^*, A_j z_y - c_{y\,j}\rangle - \langle l_{2j,q}^*, A_j z_y - c_{y\,j}\rangle) + \sum_{j=m'+1}^{m''} \langle l_{m'+j,q}^*, \hat{g}_j(z_y) - c_{y\,j}\rangle \\
&\qquad + \sum_{j=1}^{m'} (\langle l_{2j-1,q}^*, c_{y\,j} - \overline{\varphi}_j\rangle + \langle l_{2j,q}^*, \underline{\varphi}_j - c_{y\,j}\rangle) + \sum_{j=m'+1}^{m''} (\langle l_{m'+j,q}^*, c_{y\,j} - \hat{\varphi}_j\rangle \\
&\le \langle y_q^*, \hat{G}(z_y) - c_y\rangle = \langle y_q^*, y\rangle
\end{aligned}$$

Since $J(\bar{z}_q) - J(z_y) \ge J(\bar{z}_q) - J(\bar{z}) - L\|\bar{z} - z_y\|_{\hat{Z}} \ge J(\bar{z}_q) - J(\bar{z}) - L\gamma\|y\|_Y$ we obtain $\|y_q^*\|_{Y^*} \le L\gamma + r^{-1}(\epsilon_q + J(\bar{z}) - J(\bar{z}_q))$ and the assertion follows, since $\bar{z}_q \to \bar{z}$ and $\epsilon_q \to 0$ by (4.1). □

By the Alaoglu-Bourbaki-Theorem, the unit ball of Y^* is weakly-$*$ compact. Therefore, by the assumptions of Proposition 4.3, for every sequence $q^n \to \bar{q}$ the sequence $y_{q^n}^*$ has at least one weak-$*$ limit point.

Note that the condition (4.5) is a constraint qualification condition commonly used in convex programming, see e.g. [2, Chapter 2.3]. It is easy to see that condition (4.5) is fulfilled if either there is some $c \in C$ such that $c \in \operatorname{int} \hat{G}(\hat{Z}_E)$ or there is some $\tilde{z} \in \hat{Z}_E$ such that $\hat{G}(\tilde{z}) \in \operatorname{int}_Y C$, i.e. a Slater condition is fulfilled. In many cases a a combination of these special cases applies:

PROPOSITION 4.4. *Assume that the constraints can be partitioned into $C = C_1 \times C_2 \subset Y_1 \times Y_2 = Y$, $\hat{G} = (\hat{G}_1, \hat{G}_2)$, $\hat{G}_i : \hat{Z} \to Y_i$, $i = 1, 2$ such that $c_1 \in \operatorname{int} \hat{G}_1(\hat{Z}_E)$ (or more generally, $0 \in \operatorname{int}_{Y_1}(\hat{G}_1(\hat{Z}_E) - C_1)$) and $\hat{G}_1(\tilde{z}) \in C_1$, $\hat{G}_2(\tilde{z}) \in \operatorname{int}_{Y_2} C_2$ for some $c_1 \in C_1$, $\tilde{z} \in \hat{Z}_E$. Then condition (4.5) is fulfilled.*

Proof. We can find some radius $r_2 > 0$ such that the ball $B_{Y_2}(\hat{G}_2(\tilde{z}), 2r_2)$ is contained in $\operatorname{int}_{Y_2} C_2$. By continuity of G_2 we can choose $r > 0$ such that $\hat{G}_2(B_{\hat{Z}_E}(\tilde{z}, r)) \subset B_{Y_2}(\hat{G}_2(\tilde{z}), r_2)$ and by [2, Propositions 2.77, 2.79] there is some radius $r_1 > 0$ such that $B_{Y_1}(0, r_1) \subset \operatorname{int}_{Y_1} \hat{G}_1(B_{\hat{Z}_E}(\tilde{z}, r)) - C_1$. Now let $(y_1, y_2) \in B_{Y_1}(0, r_1) \times B_{Y_2}(0, r_2)$ be arbitrarily fixed. Then there is some $z \in B_{\hat{Z}_E}(\tilde{z}, r)$ such that $y_1 \in \hat{G}_1(z) - C_1$ and since $\hat{G}_2(z) \in B_{Y_2}(\hat{G}_2(\tilde{z}), r_2)$, $\|y_2\|_{Y_2} \leq r_2$ we have $\hat{G}_2(z) - y_2 \in B_{Y_2}(\hat{G}_2(\tilde{z}), 2r_2) \subset \operatorname{int}_{Y_2} C_2$ and consequently $(y_1, y_2) \in \hat{G}(z) - C$. This show $B_{Y_1}(0, r_1) \times B_{Y_2}(0, r_2) \subset \hat{G}(\hat{Z}_E) - C$ and the assertion follows. □

If we assume for a moment that J and $\hat{G}$ are continuously differentiable on $\hat{Z}$, then by the assumptions of Theorem 4.2 we have $DJ(\bar{z}) + D\hat{G}(\bar{z})^* y^* \in \hat{Z}_E^{\perp}$. If E maps $\hat{Z}$ continuously onto some Banach space $\hat{V}$ continuously embedded in V we can apply the Closed Range Theorem to obtain a first-order necessary condition of the usual form $DJ(\bar{z}) + D\hat{G}(\bar{z})^* y^* + E^* \hat{v}^* = 0_{\hat{Z}^*}$, where $\hat{v}^* \in \hat{V}^*$. In the non-differentiable but convex case we can derive a condition of the form $0_{\hat{Z}^*} \in \partial J(\bar{z}) + \partial \langle y^*, \hat{G}(\bar{z}) \rangle + E^* \hat{v}^*$ by the same assumption.

Example 1 (revisited) Let $\Omega \subset \mathbb{R}^d$ have a $C^{1,1}$ boundary and assume that the optimal solution $(\bar{y}, \bar{u})$ fulfills $\bar{u} \in L^r(\Omega)$ for some $r > d$, $r \geq 2$ (this is certainly true in case of a bilateral constraint $\underline{\varphi}_u \leq u \leq \overline{\varphi}_u$, where $\underline{\varphi}_u, \overline{\varphi}_u \in L^r(\Omega)$). By [4, Theorem 2.4.2.5] $\triangle : W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \to L^r(\Omega)$ is an isomorphism. Then we can choose $\hat{Z} = W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \times L^r(\Omega)$ and $Y = L^r(\Omega) \times C(\operatorname{cl}\Omega) \times C(\operatorname{cl}\Omega)$, since y and ∇y are continuous on $\operatorname{cl}\Omega$ for $y \in W^{2,r}(\Omega)$. The mapping $\hat{G}$ and the set C are given by

$$\hat{G}(y, u) = (u, y, |\nabla y|), \ C = C_u \times C_y \times C_g,$$

where $C_u = \{u \in L^r(\Omega) : u \leq \varphi_u\}$, $C_y = \{y \in C(\operatorname{cl}\Omega) : y \leq \varphi_y\}$ and $C_g = \{y \in C(\operatorname{cl}\Omega) : |\nabla y|_2 \leq \varphi_g\}$. By using proposition 4.4 we deduce that condition (4.5) is fulfilled in this setting, if there exists some feasible $(y, u) \in \hat{Z}$ with $y \in \operatorname{int} C_y$, $|\nabla y| \in \operatorname{int} C_g$, i.e. there is some $\delta > 0$ such that $y \leq \varphi_y - \delta$ and $|\nabla y|_2 \leq \varphi_g - \delta$ in Ω.

Example 2 Let us present an example where (4.5) holds but Proposition 4.4 does not apply. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $\hat{Z} = H_0^1(\Omega) \times L^2(\Omega)$, $E(y, u) = -\triangle y - u$, $Y = L^2(\Omega) \times L^2(\Omega)$, $\hat{G}(y, u) = (u, y + \epsilon u)$, $C = \{(v_1, v_2) \in Y : v_i \leq \varphi_i, i = 1, 2\}$, where $\epsilon > 0$, $\varphi_i \in L^2(\Omega)$. This setting occurs in the so called Lavrentiev-regularization of state constraints, see e.g. [11],[14]. We now show that condition (4.5) is fulfilled. Actually, we show that for every $(u_1, u_2) \in Y$ we can find some $(y, u) \in \hat{Z}_E$ and some $(v_1, v_2) \in C$ such that $u_1 = u - v_1$, $u_2 = y + \epsilon u - v_2$. Indeed, let $u := \min\{u_1 + \varphi_1, \frac{u_2 + \varphi_2}{\epsilon}, 0\}$ and y be the solution of $-\triangle y = u$. Since $u \leq 0$ we have $y \leq 0$ and therefore $u \leq u_1 + \varphi_1$, $y + \epsilon u \leq u_2 + \varphi_2$. By setting $v_1 = u - u_1$, $v_2 = y + \epsilon u - u_2$ the assertion follows.

5. On a superlinear convergent methods for solving the subproblems. Throughout this section let $q \in \dot{Q}$ be fixed. We now present a Newton-type method for the efficient solution of the subproblem (P_q). We suppose some additional smoothness assumptions on our problem functions:

(P5a) The objective J is twice continuously differentiable on Z_E and the second derivative $D^2 J(z)$ is bounded for z belonging to bounded subsets of Z_E.

(P5b) For each $i \in \{1, \ldots, m\}$ the mapping g_i has the representation $g_i(z)(x) = \eta_i(A_i z(x))$ for some continuous linear operator $A_i \in \mathcal{L}(Z, L^2(\Omega_i)^{d_i})$, $d_i \geq 1$ and some convex function $\eta_i : \mathbb{R}^{d_i} \to \mathbb{R}$.

Note that assumption (P5b) does not imply in general that g_i is twice continuously differentiable, even if η_i is smooth.

Further we assume that for each $i \in \{1, \ldots, m\}$ we can choose our penalty functions $\psi_{i,q}$ such that the following hypothesis is fulfilled:

(PQ2a) The mapping $\pi_i : \mathbb{R}^{d_i} \times \mathbb{R} \to \mathbb{R}$, $\pi_i(s, t) = \psi_{i,q_i}(\eta_i(s) - t)$ is twice continuously differentiable with respect to s for all $t \in \mathbb{R}$ and there is some constant L such that

$$|D^2_{ss} \pi_i(s, t)|_2 \leq L, \quad |D^2_{ss} \pi_i(s, t) - D^2_{ss} \pi_i(\bar{s}, t)|_2 \leq L |s - \bar{s}|_2 \tag{5.1}$$

for all $(s, \bar{s}, t) \in \mathbb{R}^{d_i} \times \mathbb{R}^{d_i} \times \mathbb{R}$, where

$$|D^2_{ss}\pi_i(s,t)|_2 := \sup\{\langle D^2_{ss}\pi_i(s,t)h, k\rangle \, : \, |h|_2 = 1, |k|_2 = 1\}$$

(PQ2b) $\Psi_{i,q}$ is continuous on Z.
Note that in this setting we have $\Psi_{i,q}(z) = \int_{\Omega_i} \pi_i(A_i z(x), \varphi_i(x)) \, d\mu_i(x)$.

Example 1 (revisited) (P5) is certainly fulfilled with operators $A_1(y,u) = u$, $A_2(y,u) = y$, $A_3(y,u) = \nabla y$ for the constraints on the control, the state and the gradient of the state, respectively. If we choose for our penalty functions the combined logarithmic-quadratic functions given by (2.2), then (PQ2a) is not fulfilled for the constraint $|\nabla y|_2 \le \varphi_g$. However, if we replace this constraint by the equivalent constraint

$$\hat{g}_3(y,u) := \sqrt{1 + |\nabla y|_2^2} \le \sqrt{1 + \varphi_g^2} := \hat{\varphi}_g \text{ a.e. in } \Omega$$

then assumption (PQ2) holds with $d_1 = d_2 = 1$, $d_3 = d$, $\pi_1(s,t) = \psi_{1,q}(s-t)$, $\pi_2(s,t) = \psi_{2,q}(s-t)$, $\pi_3(s,t) = \psi_{3,q}(\sqrt{1 + |s|_2^2} - t)$.

LEMMA 5.1. *For each $i \in \{1, \dots, m\}$ the mapping $\Psi_{i,q}$ is twice Gâteaux-differentiable and $\Psi_{i,q} \in C^{1,1}(Z)$, where*

$$\langle D\Psi_{i,q}(z), h\rangle = \int_{\Omega_i} \langle D_s\pi_i(A_i z(x), \varphi_i(x)), A_i h(x)\rangle \, d\mu_i(x),$$
$$\langle D^2\Psi_{i,q}(z)h, k\rangle = \int_{\Omega_i} \langle D^2_{ss}\pi_i(A_i z(x), \varphi_i(x))A_i h(x), A_i k(x)\rangle \, d\mu_i(x).$$

Further, for all $z \in Z$ we have

$$\lim_{z' \to z} \sup_{\substack{h \in \mathcal{B} \\ \tilde{k} \in \tilde{\mathcal{K}}}} \int_{\Omega_i} |\langle (D^2_{ss}\pi_i(A_i z'(x), \varphi_i(x)) - D^2_{ss}\pi_i(A_i z(x), \varphi_i(x)))h(x), \tilde{k}(x)\rangle| \, d\mu_i(x) = 0 \tag{5.2}$$

for every bounded subset $\mathcal{B} \subset L^2(\Omega_i)^{d_i}$ and whenever $\tilde{\mathcal{K}}$ is either bounded in $L^{\hat{r}_i}(\Omega_i)^{d_i}$ for some $\hat{r}_i > 2$ or the elements of $\tilde{\mathcal{K}}$ are of the form $\tilde{k}(x) = R(x)k(x)$ where k belongs to a compact subset $\mathcal{K} \subset L^2(\Omega_i)^{d_i}$ and R is from a bounded subset $\mathcal{R} \subset L^\infty(\Omega_i)^{d_i \times d_i}$.

Proof. Using similar arguments as in the proof of Lemma 3.1 we obtain

$$\Psi'_{i,q}(z,h) = \int_{\Omega_i} \langle D_s\pi_i(A_i z(x), \varphi_i(x)), A_i h(x)\rangle \, d\mu_i(x)$$

Obviously, $\Psi'_{i,q}(z,\cdot)$ is linear and it is also continuous, since $\Psi_{i,q}$ is convex and continuous. Hence, $\Psi_{i,q}$ is Gâteaux differentiable on Z. To show $\Psi_{i,q} \in C^{1,1}(Z)$, fix two points $z, \tilde{z} \in Z$ and choose $h \in Z$, $\|h\|_Z \le 1$ such that $2\langle D\Psi_{i,q}(z) - D\Psi_{i,q}(\tilde{z}), h\rangle \ge \|D\Psi_{i,q}(z) - D\Psi_{i,q}(\tilde{z})\|_{Z^*}$. Then for every $x \in \Omega_i$ there is some $\tau(x)$ in the line segment $[A_i\tilde{z}(x), A_i z(x)]$ such that

$$\begin{aligned}\langle D\Psi_{i,q}(z) - D\Psi_{i,q}(\tilde{z}), h\rangle &= \int_{\Omega_i} \langle D^2_{ss}\pi_i(\tau(x), \varphi_i(x))(A_i z(x) - A_i\tilde{z}(x)), A_i h(x)\rangle \, d\mu_i(x) \\ &\le 2L\|A_i(z - \tilde{z})\|_{L^2(\Omega_i)^{d_i}} \|A_i h\|_{L^2(\Omega_i)^{d_i}}\end{aligned}$$

and Lipschitz continuity of the derivatives follows.

Now assume that $D^2\Psi_{i,q}(z)$ is not the second Gâteaux derivative at some point $z \in Z$. Then we can find some direction $k \in Z$ and sequences $(t_n) \downarrow 0$, $(h_n) \subset Z$, $\|h_n\|_Z = 1$ such that for some positive ϵ we have

$$|\langle D\Psi_{i,q}(z + t_n k) - D\Psi_{i,q}(z) - t_n D^2\Psi_{i,q}(z)k, h_n\rangle| \ge \epsilon t_n.$$

For all n and all $x \in \Omega_i$ we can find some $\tau_n(x)$ belonging to the line segment $[A_i z(x), A_i(z+t_n k)(x)]$ such that

$$\langle D\Psi_{i,q}(z + t_n k) - D\Psi_{i,q_i}(z), h_n\rangle = t_n \int_{\Omega_i} \langle D^2_{ss}\pi_i(\tau_n(x), \varphi_i(x))A_i k(x), A_i h_n(x)\rangle \, d\mu_i(x),$$

implying

$$\epsilon \le \Big|\int_{\Omega_i} \langle (D^2_{ss}\pi_i(\tau^n(x),\varphi_i(x)) - D^2_{ss}\pi_i(A_iz(x),\varphi_i(x))A_ik(x), A_ih_n(x)\rangle\, d\mu_i(x)\Big|$$
$$\le \left(\int_{\Omega_i} |(D^2_{ss}\pi_i(\tau_n(x),\varphi_i(x)) - D^2_{ss}\pi_i(A_iz(x),\varphi_i(x))|_2^2\, |A_ik(x)|_2^2\, d\mu_i(x)\right)^{\frac12}$$
$$\cdot \|A_ih_n\|_{L^2(\Omega_i)^{d_i}}$$

for all n. However, the integrand in the last expression converges pointwise to 0 and is uniformly bounded by the integrable function $4L^2|A_ik(x)|_2^2$ and so the right hand side converges to 0 by Lebesgue's dominated convergence theorem, a contradiction. Hence, $\Psi_{i,q}$ is twice Gâteaux differentiable at z with derivative $D^2\Psi_{i,q}(z)$.

In case that $\tilde{\mathcal{K}}$ is a bounded subset of $L^{\hat r_i}(\Omega_i)^{d_i}$, $\hat r_i > 2$, let us define for every $z' \in Z$ and every $x \in \Omega_i$ the number

$$\tau_{z'}(x) = \sup_{|h|_2=1}\ \sup_{|k|_{\hat r_i}=1} \langle (D^2_{ss}\pi_i(A_iz'(x),\varphi_i(x)) - D^2_{ss}\pi_i(A_iz(x),\varphi_i(x)))h, k\rangle.$$

Then we obtain for every $h \in \mathcal{B}$ and every $\tilde k \in \tilde{\mathcal{K}}$, with $\hat r_i'^{-1} + \hat r_i^{-1} = 1$,

$$\int_{\Omega_i} |\langle (D^2_{ss}\pi_i(A_iz'(x),\varphi_i(x)) - D^2_{ss}\pi_i(A_iz(x),\varphi_i(x)))h(x), \tilde k(x)\rangle|\, d\mu_i(x)$$
$$\le \int_{\Omega_i} \tau_{z'}(x)|h(x)|_2|\tilde k(x)|_{\hat r_i}\, d\mu_i(x)$$
$$\le \left(\int_{\Omega_i} (\tau_{z'}(x)|h(x)|_2)^{\hat r_i'}\, d\mu_i(x)\right)^{\frac{1}{\hat r_i'}} \left(\int_{\Omega_i} |\tilde k(x)|_{\hat r_i}^{\hat r_i}\, d\mu_i(x)\right)^{\frac{1}{\hat r_i}}$$
$$\le \left(\int_{\Omega_i} \tau_{z'}(x)^{\frac{2}{2-\hat r_i'}}\, d\mu_i(x)\right)^{\frac{2-\hat r_i'}{\hat r_i'}} \left(\int_{\Omega_i} |h(x)|_2^2\, d\mu_i(x)\right)^{\frac12} \left(\int_{\Omega_i} |\tilde k(x)|_{\hat r_i}^{\hat r_i}\, d\mu_i(x)\right)^{\frac{1}{\hat r_i}}$$

From this inequality the equation (5.2) follows, since $\tau_{z'}$ converges pointwise to 0 on Ω_i for $z' \to z$ due to (5.1) and the equivalence of norms in $\mathbb{R}^{d_i}$ and since $\tau_{z'}$ is uniformly bounded for all z' and therefore $\int_{\Omega_i} \tau_{z'}(x)^{\frac{2}{2-\hat r_i'}}\, d\mu_i(x) \to 0$.

In the other case let us define for every $k \in L^2(\Omega_i)^{d_i}$ and every $z' \in Z$

$$\beta(z',k)$$
$$:= \sup_{\substack{h\in\mathcal{B}\\ R\in\mathcal{R}}} \int_{\Omega_i} |\langle (D^2_{ss}\pi_i(A_iz'(x),\varphi_i(x)) - D^2_{ss}\pi_i(A_iz(x),\varphi_i(x)))h(x), Rk(x)\rangle|\, d\mu_i(x)$$
$$\le br\left(\int_{\Omega_i} |D^2_{ss}\pi_i(A_iz'(x),\varphi_i(x)) - D^2_{ss}\pi_i(A_iz(x),\varphi_i(x))|_2^2\, |k(x)|_2^2\, d\mu_i(x)\right)^{\frac12},$$

where $b := \sup_{h\in\mathcal{B}}\left(\int_{\Omega_i} |h(x)|_2^2\, d\mu_i(x)\right)^{\frac12}$, $r := \sup_{R\in\mathcal{R}} \sup\{|R(x)|_2 \,:\, x \in \Omega_i\}$. Hence, similar as above, $\lim_{z'\to z}\beta(z',k) = 0$ follows by Lebesgue's dominated convergence theorem. For arbitrarily fixed $\epsilon > 0$ we can find a neighborhood U_k of z such that $0 \le \beta(z', \bar k) < \epsilon$, $\forall z' \in U_k$. Further, for all $k_1, k_2 \in L^2(\Omega_i)^{d_i}$ we have $\beta(z',k_1) - \beta(z',k_2) \le \beta(z', k_1 - k_2) \le 2Lbr\|k_1 - k_2\|_{L^2(\Omega_i)^{d_i}}$. Thus we can find some open neighborhood V_k of k such that $|\beta(z',k') - \beta(z',k)| \le \epsilon$, $\forall z'$, $\forall k' \in V_k$. The sets V_k, $k \in \mathcal{K}$ are an open covering of the compact set $\mathcal{K}$ which can be reduced to some finite covering $V_{k_1},\ldots,V_{k_N}$. Then for every z' belonging to the neighborhood $U_\epsilon := \bigcap_{j=1}^N U_{k_j}$ of z and every $k \in \mathcal{K}$ we can find some k_j with $k \in V_{k_j}$ such that $0 \le \beta(z',k) \le \beta(z',k_j) + \epsilon \le 2\epsilon$. Hence, for every $z' \in U_\epsilon$ we have $\sup_{k\in\mathcal{K}} \beta(z',k) \le 2\epsilon$ and $\lim_{z'\to z}\sup_{k\in\mathcal{K}}\beta(z',k) = 0$ together with equation (5.2) follows. □

COROLLARY 5.2. *$\Psi_{i,q}$ is twice continuously differentiable on Z_E if either A_i is a compact linear operator from Z_E into $L^2(\Omega_i)^{d_i}$ or $A_i \in \mathcal{L}(Z_E, L^{\hat{r}_i}(\Omega_i)^{d_i})$ with $\hat{r}_i > 2$.*

Note that due to Assumption (P5) J_q is at least twice Gâteaux differentiable on Z_E, the second derivatives $D^2 J_q(z)$ are bounded for bounded z and

$$\langle D^2 J_q(z)h, h\rangle \geq \langle D^2 J(z)h, h\rangle \geq \alpha \|h\|_Z^2, \forall z, h \in Z_E, \tag{5.3}$$

due to (P3) and $\langle D^2 \Psi_{i,q_i} h, h\rangle \geq 0$, $\forall i = 1, \ldots, m$. By the optimality of $\bar{z}_q$ for the problem (P_q) we have $DJ_q(\bar{z}_q) = 0$. Further, for any bounded convex neighborhood U of $\bar{z}_q$ we have

$$\frac{\alpha}{2}\|z - \bar{z}_q\|_Z^2 \leq J_q(z) - J_q(\bar{z}_q) \leq \langle DJ_q(z), z - \bar{z}_q\rangle \leq C_U \|z - \bar{z}_q\|_Z^2 \tag{5.4}$$

for every $z \in U \cap Z_E$, where $C_U := \sup\{\|D^2 J_q(z)\|_{\mathcal{L}(Z_E, Z_E^*)} \ : \ z \in U \cap Z_E\}$. This follows from the convexity of J_q and the fact that for every $z \in U \cap Z_E$ there are some $\zeta', \zeta'' \in [0,1]$ such that $J_q(z) - J_q(\bar{z}_q) = \langle DJ_q(\bar{z}_q), z - \bar{z}_q\rangle + \frac{1}{2}\langle D^2 J_q(\zeta' z_q + (1-\zeta')z)(z - \bar{z}_q), z - \bar{z}_q\rangle = \frac{1}{2}\langle D^2 J_q(\zeta' z_q + (1-\zeta')z)(z - \bar{z}_q), z - \bar{z}_q\rangle$ and $\langle DJ_q(z), z - \bar{z}_q\rangle = \langle DJ_q(z) - DJ_q(\bar{z}_q), z - \bar{z}_q\rangle = \langle D^2 J_q(\zeta'' \bar{z}_q + (1 - \zeta'')z)(z - \bar{z}_q), z - \bar{z}_q\rangle$, respectively.

Now we state some basic algorithm for solving the problem (P_q).

Algorithm 1 (Newton's method with line search):
0. Choose $0 < \gamma < 1$, $z^0 \in Z_E$.
For $n = 0, 1, \ldots$:
1. Compute $h^n \in Z_E$ as the solution of

$$[\min_{h \in Z_E} \langle DJ_q(z^n), h\rangle + \frac{1}{2}\langle D^2 J_q(z^n)h, h\rangle \tag{5.5}$$

2. Line search: Choose the step size as the maximum $\sigma_n \in \{1, \frac{1}{2}, \frac{1}{4}, \ldots\}$ for which $J_q(z^n + \sigma_n h^n) \leq J_q(z^n) + \gamma\sigma_n \langle DJ_q(z^n), h^n\rangle$.
3. Set $z^{n+1} := z^n + \sigma_n h^n$.

Note that the search direction $h^n \in Z_E$ is given by the solution of the linear system

$$D^2 J_q(z^n)h = -DJ_q(z^n)$$

which is uniquely solvable in Z_E due to (5.3) and the Lax-Milgram lemma, see e.g.[9].

THEOREM 5.3. *Assume that the assumptions (P1)–(P5,) (Q1), (Q2) and (PQ2) are fulfilled and let (z^n) be generated by Algorithm 1. Then* $\lim_{n\to\infty} z^n = \bar{z}_q$.

Proof. Since

$$\begin{aligned}\langle DJ_q(z^n), h^n\rangle &= -\langle D^2 J_q(z^n)h^n, h^n\rangle \leq -\alpha\|h_n\|_{Z_E}^2 \\ &\leq -\frac{\alpha}{\|D^2 J_q(z^n)\|_{\mathcal{L}(Z_E, Z_E^*)}}\|h^n\|_{Z_E}\|DJ_q(z^n)\|_{Z_E^*},\end{aligned}$$

$\|h^n\|_{Z_E}^2 \geq \frac{-\langle DJ_q(z^n), h^n\rangle}{\|D^2 J_q(z^n)\|_{\mathcal{L}(Z_E, Z_E^*)}}$ and $DJ_q(\cdot)$ is Lipschitz on bounded sets, the assertion follows from the general theory of descent methods, see e.g. [9, Chapter 2.2] ☐

If J_q is twice continuously differentiable on Z_E, the speed of convergence of Algorithm 1 will be q–superlinear. More precisely, there exists an increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to 0_+} \omega(t) = 0$ such that $\|z^{n+1} - \bar{z}_q\|_{Z_E} \leq \omega(\|z^n - \bar{z}_q\|_{Z_E})\|z^n - \bar{z}_q\|_{Z_E}$. Otherwise, convergence will be only q-linear, that is $0 < \lim_{t\to 0_+} \omega(t) < 1$. Note that in case of q-linear convergence it can also happen that for any starting point z^0 Algorithm 1 generates a sequence which converges q-superlinear to $\bar{z}_q$, i.e. $\limsup_{n\to\infty} \frac{\|z^{n+1} - \bar{z}_q\|_{Z_E}}{\|z^n - \bar{z}_q\|_{Z_E}} = 0$.

In what follows we consider a modification of Algorithm 1 in such a way, that we also achieve superlinear convergence in case that some of the functions $\Psi_{i,q}$ are not twice continuously differentiable on Z_E. We call this modification a *smoothed Newton step.* For the description of such a smoothed step we need some additional structure:

(P6a) There exist some integers m', $\tilde{d}$, a finite measure space $(\tilde{\Omega}_i, \tilde{\mathcal{A}}, \tilde{\mu})$, some reals γ_i, a linear operator $B \in \mathcal{L}(Z, L^2(\tilde{\Omega})^{\tilde{d}})$ and some compact linear operators $C_i \in \mathcal{L}(Z, L^2(\tilde{\Omega})^{\tilde{d}})$ such that $d_i = \tilde{d}, (\Omega_i, \mathcal{A}_i, \mu_i) = (\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$, $A_i = \gamma_i B + C_i$ for all $i = 1, \dots, m'$. Further we assume $\Psi_{i,q} \in C^2(Z_E)$, $i = m'+1, \dots, m$.

(P6b) The mapping $\mathcal{H} : Z \to V \times L^2(\tilde{\Omega})^{\tilde{d}}$, $\mathcal{H}(z) = (Ez, Bz)$ is surjective.

We denote by $\mathcal{U} \subset Z$ some topological complement space of Ker B. For each $z \in Z_E$ we denote by $D\hat{J}(z) \in Z^*$ some continuous extension of the linear $DJ(z) \in Z_E^*$ and by $D^2\hat{J}(z) \in \mathcal{L}(Z, Z^*)$ some continuous extension of $D^2J(z) \in \mathcal{L}(Z_E, Z_E^*)$ such that $\langle D^2\hat{J}(z)h, k\rangle = \langle D^2\hat{J}(z)k, h\rangle$, $\forall k, h \in Z$ and $\langle D^2\hat{J}(z)h, h\rangle \geq \alpha\|h\|_Z^2$, $\forall h \in \mathcal{U}$.

For each $z \in Z_E$ we denote by F_z and G_z the quadratic functionals on Z given by $F_z(h) := \langle D\hat{J}(z), h\rangle + \frac{1}{2}\langle D^2\hat{J}(z)h, h\rangle + \sum_{i=m'+1}^m(\langle D\Psi_{i,q}, h\rangle + \frac{1}{2}\langle D^2\Psi_{i,q}(z)h, h\rangle)$ and $G_z(h) := F_z(h) + \sum_{i=1}^{m'}(\langle D\Psi_{i,q}, h\rangle + \frac{1}{2}\langle D^2\Psi_{i,q}(z)h, h\rangle)$. Note that we have $\langle DG_z(0), h\rangle = \langle DJ_q(z), h\rangle$, $\langle D^2G_z(0)h, k\rangle = \langle D^2J_q(z)h, k\rangle$, $\forall h, k \in Z_E$.

The smoothed Newton step consists of the following steps

1. Given the iterate z^n, compute a solution $\zeta_1 \in Z_E$ of the quadratic problem

$$\min_\zeta G_{z^n}(\zeta) \text{ s.t. } \mathcal{H}\zeta = 0$$

together with a multiplier $(v_1^*, \nu^*) \in V^* \times (L^2(\tilde{\Omega})^{\tilde{d}})^*$ such that

$$DG_{z^n}(\zeta_1) + E^*v_1^* + B^*\nu^* = 0$$

in Z^*.

2. Compute a solution $\zeta_2 \in \mathcal{U}$ of the nonlinear problem

$$\begin{aligned}\min_{\zeta\in\mathcal{U}} T_n(\zeta) := \langle E^*v_1^*, \zeta\rangle + F_{z^n}(\zeta_1 + \zeta) + \sum_{i=1}^{m'} \int_{\tilde{\Omega}} \Big(&\pi_i((A_i\tilde{z}^n + \gamma_i B\zeta)(x), \varphi_i(x)) \\ &+\langle D_s\pi_i(A_i\tilde{z}^n(x), \varphi_i(x)), C_i\zeta(x)\rangle\Big)\, d\tilde{\mu}(x),\end{aligned}$$

where $\tilde{z}^n = z^n + \zeta_1$.

3. Compute a solution $\zeta_3 \in Z$ of the quadratic problem

$$\begin{aligned}\min_\zeta S_n(\zeta) &:= F_{z^n}(\zeta_1 + \zeta_2 + \zeta) + \sum_{i=1}^{m'}\langle D\Psi_{i,q}(\hat{z}^n), \zeta\rangle + \frac{1}{2}\langle D^2\Psi_{i,q}(\hat{z}^n)\zeta, \zeta\rangle \\ &\text{s.t. } E(\zeta_2 + \zeta) = 0.\end{aligned}$$

where $\hat{z}^n = z^n + \zeta_1 + \zeta_2$. We denote the corresponding multiplier with $v_3^* \in V^*$.

4. Set $z^{n+1} = z^n + \zeta_1 + \zeta_2 + \zeta_3$.

LEMMA 5.4. $\|\zeta_1\|_Z + \|\zeta_2\|_Z + \|\zeta_3\|_Z + \|\nu^*\|_{L^2(\tilde{\Omega})^{\tilde{d}}} + \|v_3^* - v_1^*\|_{V^*} = \mathcal{O}(\|z^n - \bar{z}_q\|_{Z_E})$.

Proof. Due to Assumption (P5a) and Lemma 5.1 there exists some constant L such that $|\langle DG_{z^n}(0), h\rangle| = |\langle DJ_q(z^n), h\rangle| = |\langle DJ_q(z^n) - DJ_q(\bar{z}_q), h\rangle| \leq L\|z^n - \bar{z}_q\|_{Z_E}\|h\|_{Z_E}$, $\forall h \in Z_E$ if $\|z_n - \bar{z}_q\|_{Z_E} \leq 1$. Hence, as a consequence of the Lax-Milgram Lemma we have $\|\zeta_1\|_Z = \|\zeta_1\|_{Z_E} \leq \frac{L}{\alpha}\|z^n - \bar{z}_q\|_{Z_E}$. Using the open mapping theorem we conclude from Assumption (P6b) that there is some constant C such that for every $\zeta \in \mathcal{U}$ there is some $\hat{\zeta} \in \operatorname{Ker} B$ fulfilling $E\hat{\zeta} = -E\zeta$ and $\|\hat{\zeta}\|_Z \leq C\|E\zeta\|_V \leq C\|E\|_{\mathcal{L}(Z,V)}\|\zeta\|_Z$. Thus for all $\zeta \in \mathcal{U}$ we have

$$\begin{aligned}(5.6)\qquad |\langle B^*\nu^*, \zeta\rangle| &= |\langle B^*\nu^*, \zeta + \hat{\zeta}\rangle| = |\langle DG_{z^n}(\zeta_1), \zeta + \hat{\zeta}\rangle + \langle v^*, E(\zeta + \hat{\zeta})\rangle| \\ &\leq |\langle DG_{z^n}(0), \zeta + \hat{\zeta}\rangle| + |\langle D^2G_{z^n}(0)\zeta_1, \zeta + \hat{\zeta}\rangle| \\ &= \mathcal{O}(\|z^n - \bar{z}_q\|_{Z_E}\|\zeta + \hat{\zeta}\|_{Z_E}) = \mathcal{O}(\|z^n - \bar{z}_q\|_{Z_E}\|\zeta\|_Z)\end{aligned}$$

Then we can proceed to obtain

$$\begin{aligned}\langle DT_n(0),\zeta\rangle &= \langle E^*v^*,\zeta\rangle + \langle DF_{z^n}(\zeta_1),\zeta\rangle + \sum_{i=1}^{m'}\int_{\tilde\Omega}\langle D\pi_i(A_i\tilde z^n(x),\varphi_i(x)),A_i\zeta(x)\rangle\, d\tilde\mu(x)\\ &= \langle E^*v^*,\zeta\rangle + \langle DG_{z^n}(\zeta_1),\zeta\rangle + \sum_{i=1}^{m'} r_i^n(\zeta)\\ &= -\langle B^*\nu^*,\zeta\rangle + \sum_{i=1}^{m'} r_i^n(\zeta) = \mathcal{O}(\|z^n-\bar z_q\|_{Z_E}\|\zeta\|_Z)\end{aligned}$$

where

$$\begin{aligned}r_i^n(\zeta) := \int_{\tilde\Omega}&\langle D_s\pi_i(A_i\tilde z^n(x),\varphi_i(x)) - D_s\pi_i(A_iz^n(x),\varphi_i(x)),A_i\zeta(x)\rangle\\ &-\langle D^2_{ss}\pi_i(A_iz^n(x),\varphi_i(x))A_i\zeta_1(x),A_i\zeta(x)\rangle\, d\tilde\mu(x) = \mathcal{O}(\|\zeta_1\|_Z\|\zeta\|_Z).\end{aligned}$$

Further, T_n is twice Gâteaux differentiable on $\mathcal{U}$ and $\langle D^2T_n(\zeta)h,h\rangle \ge \alpha\|h\|_Z^2$, $\forall\zeta,h\in\mathcal{U}$. Hence we obtain $\mathcal{O}(\|z^n-\bar z_q\|_{Z_E}\|\zeta_2\|_Z) = \langle -DT_n(0),\zeta_2\rangle = \langle DT_n(\zeta_2)-DT_n(0),\zeta_2\rangle = \langle D^2T_n(\xi\zeta_2)\zeta_2,\zeta_2\rangle \ge \alpha\|\zeta_2\|_Z^2$, where $\xi\in[0,1]$ and consequently $\|\zeta_2\|_Z = \mathcal{O}(\|z^n-\bar z_q\|_{Z_E})$ follows. Obviously, ζ_3 can be written in the form $\zeta_3 = -\zeta_2+\hat\zeta_3$, where $\hat\zeta_3\in Z_E$. For all $h\in Z_E$ we have

$$\begin{aligned}&\langle DS_n(-\zeta_2),h\rangle\\ &= \langle DF_{z^n}(\zeta_1) + \sum_{i=1}^{m'}(D\Psi_{i,q}(\hat z^n) + D^2\Psi_{i,q}(\hat z^n)(-\zeta_2)),h\rangle\\ &= \langle DG_{z^n}(0) + D^2G_{z^n}(0)\zeta_1,h\rangle\\ &\qquad + \sum_{i=1}^{m'}\langle D\Psi_{i,q}(\hat z^n) + D^2\Psi_{i,q}(\hat z^n)(-\zeta_2)) - D\Psi_{i,q}(z^n) - D^2\Psi_{i,q}(z^n)\zeta_1,h\rangle\\ &= \mathcal{O}(\|z^n-\bar z_q\|_{Z_E}\|h\|_{Z_E}).\end{aligned}$$

This implies

$$\mathcal{O}(\|z^n-\bar z_q\|_{Z_E}\|\hat\zeta_3\|_{Z_E}) = \langle DS_n(-\zeta_2),\hat\zeta_3\rangle = \langle DS_n(\zeta_3)-DS_n(-\zeta_2),\hat\zeta_3\rangle = \langle D^2S_n(0)\hat\zeta_3,\hat\zeta_3\rangle \ge \frac{\alpha}{2}\|\hat\zeta_3\|_{Z_E}^2$$

and therefore $\|\hat\zeta_3\|_{Z_E} = \mathcal{O}(\|z^n-\bar z_q\|_{Z_E})$ holds.

Since $\langle B^*\nu^*,\zeta\rangle = 0$, $\zeta\in\operatorname{Ker}B$ and Z is the topological direct sum of $\mathcal{U}$ and $\operatorname{Ker}B$, we obtain from (5.6) the estimate $\|B^*\nu^*\|_{Z^*} = \mathcal{O}(\|z^n-\bar z_q\|_{Z_E}$. By Assumption (P6b) B is surjective and hence B^* is a homeomorphism from $(L^2(\tilde\Omega)^{\tilde d})^*$ onto the range of B^* and $\|\nu^*\|_{L^2(\tilde\Omega)^{\tilde d}} = \mathcal{O}(\|z^n - \bar z_q\|_{Z_E}$ follows. Finally we have

$$\begin{aligned}E^*(v_3^*-v_1^*) &= DG_{z^n}(\zeta_1) + B^*\nu^* - DS_n(\zeta_3)\\ &= B^*\nu^* + D^2F_{z^n}(0)(\zeta_1-\zeta_3)\\ &\qquad + \sum_{i=1}^{m'}(D\Psi_{i,q}(z^n) - D\Psi_{i,q}(\hat z^n) + D^2\Psi_{i,q}(z^n)\zeta_1 - D^2\Psi_{i,q}(\hat z^n)\zeta_3)\\ &= \mathcal{O}(\|z^n-\bar z_q\|_{Z_E}\end{aligned}$$

and since E is surjective we obtain $\|v_3^*-v_1^*\|_{V^*} = \mathcal{O}(\|z^n-\bar z_q\|_{Z_E}$. □

THEOREM 5.5. *Assume that the assumptions (P1)–(P6,) (Q1), (Q2) and (PQ2) are fulfilled and assume that there is some bounded set $\tilde{\mathcal{K}}\subset L^2(\tilde\Omega)^{\tilde d}$, which is either bounded in $L^{\hat r}(\tilde\Omega)^{\tilde d}$, $\hat r>2$ or the elements are of the form $\tilde k(x) = R(x)k(x)$ where k belongs to a compact subset $\mathcal{K}\subset L^2(\tilde\Omega)^{\tilde d}$*

and R is from a bounded subset $\mathcal{R} \subset L^\infty(\tilde\Omega)^{\tilde d\times\tilde d}$, such that for the smoothed Newton steps we have $\operatorname{dist}(B\zeta_3, \|B\zeta_3\|\tilde{\mathcal{K}}) = o(\|z^n - \bar z_q\|_Z)$ *for all $z^n \in Z_E$ in some neighborhood of $\bar z_q$. Then there exists a increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$, $\lim_{t\to 0_+}\omega(t) = 0$ such that $\|z^{n+1} - \bar z_q\|_Z \le \omega(\|z^n - \bar z_q\|_Z)\|z^n - \bar z_q\|_Z$, where z^{n+1} denotes the outcome of the smoothed Newton step.*

Proof. Because J and $\Psi_{i,q_i}, i = m'+1,\ldots,m$ are twice continuously differentiable on Z_E and optimality of ζ_3 implies the first-order necessary condition $\langle DS_n(\zeta_3), h\rangle = 0$, $\forall h \in Z_E$, we have

$$\begin{aligned}
&\langle DJ_q(z^{n+1}), z^{n+1} - \bar z_q\rangle\\
&= \langle DJ(z^{n+1}) - DJ(z^n) - D^2J(z^n)(z^{n+1} - z^n), z^{n+1} - \bar z_q\rangle\\
&\quad + \sum_{i=m'+1}^{m} \langle D\Psi_{i,q_i}(z^{n+1}) - D\Psi_{i,q_i}(z^n) - D^2\Psi_{i,q_i}(z^n)(z^{n+1} - z^n), z^{n+1} - \bar z_q\rangle\\
&\quad + \sum_{i=1}^{m'}\langle D\Psi_{i,q_i}(z^{n+1}) - D\Psi_{i,q_i}(\hat z^n) - D^2\Psi_{i,q_i}(\hat z^n)\zeta_3, z^{n+1} - \bar z_q\rangle\\
&\quad +\langle DS_n(\zeta_3), z^{n+1} - \bar z_q\rangle\\
&= \langle (D^2J(\xi^n) - D^2J(z^n) + \sum_{i=m'+1}^{m} (D^2\Psi_{i,q_i}(\xi^n) - D^2\Psi_{i,q_i}(z^n)))(z^{n+1} - z^n), z^{n+1} - \bar z_q\rangle\\
&\quad + \sum_{i=1}^{m'}\langle (D^2\Psi_{i,q_i}(\hat\xi^n) - D^2\Psi_{i,q_i}(\hat z^n))\zeta_3, z^{n+1} - \bar z_q\rangle\\
&= \sum_{i=1}^{m'}\langle (D^2\Psi_{i,q_i}(\hat\xi^n) - D^2\Psi_{i,q_i}(\hat z^n))\zeta_3, z^{n+1} - \bar z_q\rangle\\
&\quad +o((\|z^{n+1} - \bar z_q\|_Z + \|z^n - \bar z_q\|_Z)\|z^n - \bar z_q\|_Z),
\end{aligned}$$

where $\xi^n = \tau^n z^{n+1} + (1-\tau^n)z^n$, $\hat\xi^n = \tau^n z^{n+1} + (1-\tau^n)\hat z^n$ for some $\tau^n \in [0,1]$. From (5.2) we obtain for $i = 1,\ldots,m'$

$$\begin{aligned}
&\langle (D^2\Psi_{i,q_i}(\hat\xi^n) - D^2\Psi_{i,q_i}(\hat z^n))\zeta_3, z^{n+1} - \bar z_q\rangle\\
&= \int_{\tilde\Omega} \langle P_i(x)A_i\zeta_3(x), A_i(z^{n+1} - \bar z_q)(x)\rangle\, d\mu_i(x)\\
&= \int_{\tilde\Omega} (\gamma_i\langle P_i(x)B\zeta_3(x), A_i(z^{n+1} - \bar z_q)(x)\rangle + \langle P_i(x)C_i\zeta_3(x), A_i(z^{n+1} - \bar z_q)(x)\rangle)\, d\tilde\mu(x)\\
&\le \|A_i(z^{n+1} - \bar z_q)\|_{L^2(\tilde\Omega)^{\tilde d}} \sup_{\|h\|_{L^2(\tilde\Omega)^{\tilde d}}\le 1} \Bigg(\|B\zeta_3\|_{L^2(\tilde\Omega)^{\tilde d}} \sup_{\tilde k\in\tilde{\mathcal{K}}} \int_{\tilde\Omega} \gamma_i\langle P_i(x)\tilde k(x), h(x)\rangle\, d\tilde\mu(x)\\
&\quad +\|\zeta_3\|_Z \sup_{k\in B_Z(0,1)} \int_{\tilde\Omega} \langle P_i(x)C_ik(x), h(x)\rangle\, d\tilde\mu(x) + o(\|z^n - \bar z_q\|_Z)\Bigg)\\
&= o((\|\zeta_3\|_Z + \|z^n - \bar z_q\|_Z)\|z^{n+1} - \bar z_q(x)\|_Z)
\end{aligned}$$

where

$$\begin{aligned}
P_i(x) &:= D^2\pi_i(A_i\hat\xi^n(x), \varphi_i(x)) - D^2\pi_i(A_iz^n(x), \varphi_i(x))\\
&= (D^2\pi_i(A_i\hat\xi^n(x), \varphi_i(x)) - D^2\pi_i(A_i\bar z_q(x), \varphi_i(x)))\\
&\quad -(D^2\pi_i(A_iz^n(x), \varphi_i(x)) - D^2\pi_i(A_i\bar z_q(x), \varphi_i(x))).
\end{aligned}$$

Hence we obtain $\langle DJ_q(z^{n+1}), z^{n+1} - z_q\rangle = o(\|z^n - \bar z_q\|_Z^2)$ and the assertion follows from (5.4) □

A rigorous analysis when the hypothesis of Theorem 5.5 is fulfilled goes beyond the scope of this paper. Instead we are content with a detailed view of the smoothed Newton step in case of our example.

Example 1 (revisited) Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, the operator $A_2(y,u)=y$ is certainly compact. Now let us consider the constraint on the gradient of the state. The operator $-\triangle : H_0^1(\Omega)\to H^{-1}(\Omega)$ is a homeomorphism and therefore we have $\nabla\circ(-\triangle)^{-1}\in \mathcal{L}(H^{-1}(\Omega),L^2(\Omega)^d)$. Since $L^2(\Omega)$ is compactly embedded in $H^{-1}(\Omega)$, the operator $\nabla\circ(-\triangle)^{-1}$ is compact on $L^2(\Omega)$ and hence the operator $A_3(y,u)=\nabla y$ is compact on Z_E. Consequently the mappings $\Psi_{2,q}$ and $\Psi_{3,q}$ are twice continuously differentiable on Z_E by Corollary 5.2. However, for the operator $A_1(y,u)=u$ the assumptions of Corollary 5.2 are not fulfilled and in fact one can show that the mapping $\Psi_{1,q}$ is nowhere twice Fréchet differentiable.

Let us consider the smoothed Newton-step at the iterate $z^n=(y^n,u^n)$ with $m'=1$, $B(y,u)=A_1(y,u)=u$, $\gamma_1=1$, $C_1(y,u)=0$, $\mathcal{U}=\{0\}\times L^2(\Omega)$. In the first step we have to find a solution $\zeta_1=(\zeta_{1,y},\zeta_{1,u})$ of the problem

$$\begin{aligned}\min_{\zeta_y,\zeta_u}\int_\Omega\Big(&(y^n-y_d)\zeta_y+\frac12\zeta_y^2+\beta((u^n-u_d)\zeta_u+\frac12\zeta_u^2)\\&+D_s\pi_1(u^n,\varphi_u)\zeta_u+\frac12D^2_{ss}\pi_1(u^n,\varphi_u)\zeta_u^2+D_s\pi_2(y^n,\varphi_y)\zeta_y+\frac12D^2_{ss}\pi_2(y^n,\varphi_y)\zeta_y^2\\&+\langle D_s\pi_3(\nabla y,\hat\varphi_g),\nabla\zeta_y\rangle+\frac12\langle D^2_{ss}\pi_3(\nabla y^n,\hat\varphi_g)\nabla\zeta_y,\nabla\zeta_y\rangle\Big)\\ \text{s.t.}&-\triangle\zeta_y=\zeta_u \text{ in } \Omega,\ \zeta_y=0 \text{ on } \partial\Omega\\&\zeta_u=0\text{ in }\Omega\end{aligned}$$

Obviously, the solution is $\zeta_1=(0,0)$ and the multiplier $v_1^*\in H_0^1(\Omega)$ is given by the unique solution of the variational problem

$$\int_\Omega(\langle\nabla v_1^*,\nabla v\rangle+(y^n-y_d)v+D_s\pi_2(y^n,\varphi_y)v+\langle D_s\pi_3(\nabla y^n,\hat\varphi_g),\nabla v\rangle)=0,\ \forall v\in H_0^1(\Omega)$$

In the second step we have to find $\zeta_2=(\zeta_{2,y},\zeta_{2,u})$, where $\zeta_{2,y}=0$ and $\zeta_{2,u}\in L^2(\Omega)$ is the solution of the nonlinear problem

$$\min_{\zeta_u}\int_\Omega\Big(-v_1^*\zeta_u+\beta((u^n-u_d)\zeta_u+\frac12\zeta_u^2)+\pi_1(u^n+\zeta_u,\varphi_u)\Big),$$

which can be solved pointwise, i.e. for every $x\in\Omega$ we can compute $\zeta_{2,u}(x)$ as the solution of the one-dimensional convex problem

$$\min_{\zeta\in\mathbb{R}}-v_1^*(x)\zeta+\beta((u^n(x)-u_d(x)))\zeta+\frac12\zeta^2)+\pi_1(u^n(x)+\zeta,\varphi_u(x)). \tag{5.7}$$

Finally, the solution $\zeta_3=(\zeta_{3,y},\zeta_{3,u})$ is the solution of the problem

$$\begin{aligned}\min_{\zeta_y,\zeta_u}\int_\Omega\Big(&(y^n-y_d)\zeta_y+\frac12\zeta_y^2+\beta((u^n-u_d)(\zeta_u+\zeta_{2,u})+\frac12(\zeta_u+\zeta_{2,u})^2)\\&+D_s\pi_1(u^n+\zeta_{2,u},\varphi_u)\zeta_u+\frac12D^2_{ss}\pi_1(u^n+\zeta_{2,u},\varphi_u)\zeta_u^2\\&+D_s\pi_2(y^n,\varphi_y)\zeta_y+\frac12D^2_{ss}\pi_2(y^n,\varphi_y)\zeta_y^2\\&+\langle D_s\pi_3(\nabla y^n,\hat\varphi_g),\nabla\zeta_y\rangle+\frac12\langle D^2_{ss}\pi_3(\nabla y^n,\hat\varphi_g)\nabla\zeta_y,\nabla\zeta_y\rangle\Big)\\ \text{s.t.}&-\triangle\zeta_y=\zeta_u+\zeta_{2,u}\text{ in }\Omega,\ \zeta_y=0\text{ on }\partial\Omega\end{aligned}$$

The corresponding multiplier $v_3^*\in H_0^1(\Omega)$ fulfills

$$\begin{aligned}0=\int_\Omega\big(&\langle\nabla v_3^*,\nabla v\rangle+(y^n-y_d+\zeta_{3,y}+D_s\pi_2(y^n,\varphi_y)+D^2_{ss}\pi_2(y^n,\varphi_y)\zeta_{3,y})v\\&+\langle D_s\pi_3(\nabla y^n,\hat\varphi_g)+D^2_{ss}\pi_3(\nabla y^n,\hat\varphi_g)\nabla\zeta_{3,y},\nabla v\rangle\big),\ \forall v\in H_0^1(\Omega),\\0=\beta(&u^n-u_d+\zeta_{2,u}+\zeta_{3,u})+D_s\pi_1(u^n+\zeta_{2,u},\varphi_u)+D^2_{ss}\pi_1(u^n+\zeta_{2,u},\varphi_u)\zeta_{3,u}-v_3^*.\end{aligned}$$

From equation (5.7) we deduce

$$-v_1^* + \beta(u^n - u_d + \zeta_{2,u}) + D_s\pi_1(u^n + \zeta_{2,u}, \varphi_u) = 0 \text{ in } \Omega$$

and therefore

$$(\beta + D^2_{ss}\pi_1(u^n(x) + \zeta_{2,u}(x), \varphi_u(x)))\zeta_{3,u}(x) = v_3^*(x) - v_1^*(x),\ x \in \Omega.$$

By convexity of $\pi_1(s,t)$ with respect to s we have $D^2_{ss}\pi_1(u^n(x) + \zeta_{2,u}(x), \varphi_u(x)) \geq 0$ and since $H^1_0(\Omega)$ is compactly embedded in $L^2(\Omega)$, together with Lemma 5.4, the assumptions of Theorem 5.5 are satisfied. Hence the smoothed Newton step yields superlinear convergence.

To ensure global convergence, an easy implementable possibility is to perform an alternating sequence of ordinary Newton steps with line search and smoothed Newton steps. The smoothed Newton step is only accepted, when a decrease in the objective is achieved.

6. Numerical results. We conclude the paper by a demonstration of the practical performance of our method. We consider the following two problems:

Problem 1: The setting for this problem corresponds to Example 1 with $\Omega = (0,1)^2$, $u_d \equiv 0$, $y_d(x_1,x_2) = \sin(2\pi x_1)\exp(6x_2)$, $\beta = 0.01$, $\varphi_u = 50 + |\cos(2\pi x_1)|$, $\varphi_y(x_1,x_2) = 0.5(1 + 0.25|0.5 - x_1|)$, $\varphi_g \equiv 10$.

Problem 2: The data are the same as in Problem 1, except for the bound on the gradient we choose

$$\varphi_g(x_1,x_2) = 10\min\{1, 10(|x_1 + x_2 - 1| + 20\max\{x_1 - 0.8, 0\} + 30\max\{0.6 - x_1, 0\})\}$$

Note that this bound on the gradient vanishes on the line segment connecting the points $(0.6, 0.4)$ and $(0.8, 0.2)$ and hence a Slater condition fails to hold.

For both problems we replace the constraint $|\nabla y|_2 \leq \varphi_g$ by the equivalent constraint

$$\sqrt{0.01 + |\nabla y|_2^2} \leq \sqrt{0.01 + \varphi_g^2}.$$

Our parameter space Q consists of 4 positive parameters $q = (\kappa, \epsilon_u, \epsilon_y, \epsilon_g)$. For each of the penalty functions $\psi_{1,q}$, $\psi_{2,q}$ and $\psi_{3,q}$ we choose the logarithmic-quadratic function given by (2.2) with parameters (κ, ϵ_u), (κ, ϵ_y), (κ, ϵ_g), respectively.

The discretization is done by linear finite elements on a uniform triangular grid with mesh size h. With each point $z = (y,u)$ we associate the values

$$\tau_J(z) := 10^4 h^{-2} \frac{\sum_{i=3}^m \int_\Omega \psi^\#_{i,q}(g_i(z)(x) - \varphi_i(x), -(g_i(z)(x) - \varphi_i(x)))\ dx}{\sqrt{10^{-8} + \|y\|^2_{H^1(\Omega)} + \|u\|^2_{L^2(\Omega)}}},$$

$$\tau_y(z) = 100h^{-2}\frac{\|(y-\varphi_y)^+\|_{L^\infty(\Omega)}}{10^{-4} + \|y\|_{H^1(\Omega)}},\ \tau_u(z) = 100h^{-2}\frac{\|(u-\varphi_u)^+\|_{L^2(\Omega)}}{10^{-4} + \|u\|_{L^2(\Omega)}},$$

$$\tau_g(z) = 100h^{-2}\frac{\|(\sqrt{0.01 + |\nabla y|^2} - \sqrt{0.01 + \varphi_g^2})^+\|_{L^\infty(\Omega)}}{10^{-4} + \|\nabla y\|_{L^2(\Omega)^2}},\ \tau(z) = \max\{\tau_J(z), \tau_u(z), \tau_y(z), \tau_g(z)$$

where $f^+(x) := \max\{f(x), 0\}$ for a function $f : \Omega \to \mathbb{R}$. We start our algorithm with parameters $\kappa^1 = 10^{-2}h^2$, $\epsilon^1_y = \epsilon^1_g = \sqrt(\kappa)$, $\epsilon_u = \sqrt{\kappa/\beta}$ and the solution of the quadratic problem

$$\begin{aligned} \min_{z=(y,u)} J(z) &= \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|u - u_d\|^2_{L^2(\Omega)} \\ \text{subject to } -\triangle y &= u \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The n-th iterate of Newton's method with line search for solving problem (P_{q^k}) is denoted by $z^{k,n}$. In our model (5.5) the expected decrease in the objective function for a step length $\sigma = 1$ is $\xi^{k,n} = \frac{1}{2}\langle DJ_{q^k}(z^{k,n}, h^{k,n}\rangle$. We accept the point $z^{k,n+1}$ as approximate solution of the problem (P_{q^k}), if $|\xi^{k,n}| \leq (10^{-3} \min\{\frac{\tau(z^{k,n+1}-1}{10}, 1\}^3 + 10^{-14}(1 - \min\{\frac{\tau(z^{k,n+1}-1}{10}, 1\}^3)(1 + |J_{q^k}(z^{k,n+1})$. We perform a smoothed Newton step only after every 4 iterations, when the ratio between the L^∞ and the L^2 norm of the part of Newton's direction corresponding to the control u exceeds 10. The point $z^{k,n+1}$ is accepted as solution of the overall problem, if $\tau(z^{k,n+1}) \leq 1$, i.e. the required accuracy behaves like $O(h^2)$. However, if $\tau(z^{k,n+1}) > 1$, we compute a new parameter vector q^{k+1} according to

$$\kappa^{k+1} = \kappa^k/\theta(\tau_J(z^{k,n+1}), m^k),\ \epsilon_u^{k+1} = \epsilon_u^k/\theta(\tau_u(z^{k,n+1}), m^k),$$

$$\epsilon_y^{k+1} = \epsilon_y^k/\theta(\tau_y(z^{k,n+1}), m^k), \epsilon_g^{k+1} = \epsilon_g^k/\theta(\tau_g(z^{k,n+1}), m^k),$$

where $\theta(\tau, m) = \max\{\min\{1.1\tau,\ m\},\ 1.05\}$. The factor m^k is given by the formula $m^k = \max\{1.2, \min\{2, 1 + \frac{m^{k-1}-1}{\min\{2,\max\{0.5,\delta^k\}\}}\}$, where

$$\delta^k = \max\{\frac{|J_{q^k}(z^{k,1} + h^{k,1}) - J_{q^k}(z^{k,1}) - \xi^{k,1}|}{0.1|\xi^{k,1}|}, \frac{|\xi^{k,1}|}{0.01(1 + |J_{q^k}(z^{k,1})|)}\}$$

measures the quality of the first Newton step when solving the k-th subproblem. As starting point for the new subproblem we use the outcome of the previous one. Using this strategy we usually need only 1 or 2 Newton steps for approximately solving each subproblem with exception of the first and last one.

In Table 1 we give the total number of required Newton steps for different grid sizes h. The numbers in parenthesis are the number of smoothed Newton steps performed by the algorithm. We observe a slight dependence of the iteration numbers with the grid size according to the increasing accuracy.

h	Problem 1	Problem 2
$\frac{1}{16}$	31 (0)	37 (0)
$\frac{1}{32}$	38 (0)	40 (0)
$\frac{1}{64}$	46 (0)	51 (1)
$\frac{1}{128}$	44 (1)	65 (1)
$\frac{1}{256}$	48 (1)	76 (2)

Table 1:Number of Newton steps (smoothed Newton steps) for various grid sizes

In order to demonstrate the effect of a smoothed Newton step we present some results when solving the last subproblem for the Problem 1 with $h = \frac{1}{128}$. h_y respectively h_u denote the part of the Newton direction corresponding to the state and the control, respectively. For the first 4 iterations normal Newton steps were performed, which seem to converge only linearly. At the 5-th iteration one smoothed Newton step is performed, which produces a very accurate approximation of the solution and after one further Newton step the algorithm terminates.

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It.	$\|h_y\|_{L^\infty(\Omega)}$	$\|h_u\|_{L^\infty(\Omega)}$	$\|h_u\|_{L^2(\Omega)}$
1	1.1E-4	2.2E0	6.8E-2
2	5.7E-5	1.3E0	4.0E-2
3	1.0E-5	6.8E-1	7.8E-3
4	3.1E-6	3.2E-1	4.2E-3
5*	2.7E-6	1.9E-1	3.6E-3
6	7.4E-14	6.7E-8	5.2E-9

* smoothed Newton step

Table 2: Iterates for the last subproblem of Problem 1 ($h = \frac{1}{128}$)

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